JOURNAL OF FUNCTIONAL ANALYSIS 96, 219–255 (1991)

On the Connection between the Malliavin Covariance Matrix and Hörmander's Condition

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Communicated by the Editors

Received July 1989

A celebrated theorem of Hörmander gives a sufficient condition for a second order differential operator to be hypoelliptic. For operators with analytic coefficients this condition turns out to be also necessary but this is not true for general smooth coefficients. On the other hand Malliavin conceived a probabilistic approach to the same problem, known as "Malliavin calculus," in which a key role is played by the "Malliavin covariance matrix." The aim of our paper is to give several characterizations of the Malliavin covariance matrix which are equivalent to Hörmander's condition (and consequently imply the hypoellipticity). In this way the distance between Hörmander's condition and the hypoellipticity property is clearly pointed out in probabilistic terms. © 1991 Academic Press, Inc.

INTRODUCTION

Let $B=(B^1, ..., B^d)$ be a Brownian motion on some probability space $(\Omega, \mathscr{F}, P)$, $\varphi_j \in C_b(R^n, R^n)$, $0 \leqslant j \leqslant d$, and the stochastic equation

$$dX(t)=\sum_{j=1}^{d} \varphi_j(X(t)) \circ dB_t^j+\varphi_0(X(t))\, dt, \qquad X(0, x, \omega)=x.$$

Malliavin calculus produces sufficient conditions in order that $P \circ X(t, x, \cdot)^{-1}$ has a smooth density with respect to the n-dimensional Lebesgue measure. The key step in this procedure is to prove that $1/\det \sigma(t)$ has finite moments of any order, where $\sigma(t)$ denotes in Malliavin calculus the covariance matrix associated with the functional $\omega \rightarrow X(t, x, \omega)$. In a first stage one verifies that σ fulfils a certain stochastic equation and then, by applying the variance 0 constants method one writes σ in the form

$$\sigma(t)=Y(t)\left[\int_0^t (Z_s \varphi(X_s))(Z_s \varphi(X_s))^{\wedge}\, ds\right] \hat{Y}(s),$$

0022-1236/91 $3.00

where $\varphi = (\varphi_1, ..., \varphi_d)$, $\hat{Y}$ designates the transposed matrix, and Y and Z are the solutions of the stochastic equations

$$dY_t = \sum_{j=1}^{d} \dot{\varphi}_j(X_t)\, Y_t \circ dB_t^j + \dot{\varphi}_0(X_t)\, Y_t\, dt, \qquad Y_0 = I,$$

and

$$dZ_t = -\sum_{j=1}^{d} Z_t \dot{\varphi}_j(X_t) \circ dB_t^j - Z_t \dot{\varphi}_0(X_t)\, dt \qquad Z_0 = I.$$

In the above equations $\dot{\varphi}_j$ is the Jacobian matrix attached to φ_j and I is the identity matrix.

Since $1/\det Y(t) = \det Z(t)$ has finite moments of any order, the problem is to show that

$$E\left(1/\det\left(\int_0^t U_x(s)\, \hat{U}_x(s)\, ds\right)^p\right) < \infty \qquad \text{for every} \quad p \in N, \tag{a}$$

where $U_x(s) = Z(s)\, \varphi(X(s, x, \omega))$.

It has been proved that under Hörmander's condition (a) holds and consequently the above-mentioned absolute continuity problem is solved.

In our frame Hörmander's condition may be stated as follows. Let $\mathscr{D}_0 = \operatorname{Span}\{\varphi_1, ..., \varphi_d\}$, $\mathscr{D}_{k+1} = \operatorname{Span}\{[\varphi_j, f],\ 0 \leqslant j \leqslant d,\ f \in \mathscr{D}_k\}$, $\mathscr{D}_\infty = \bigcup_{k=0}^{\infty} \mathscr{D}_k$, and $\mathscr{D}_{\infty, x} = \{f(x) : f \in \mathscr{D}_\infty\}$, where $[\ ,\ ]$ designates the Lie bracket (note that the drift coefficient φ_0 does not appear in $\mathscr{D}_0$ but only in $\mathscr{D}_k$, $k \geqslant 1$). Then the condition is

$$\dim \mathscr{D}_{\infty, x} = n. \tag{H$_x$}$$

We shall deal with (a) in the general frame we present below. On the space $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t \geqslant 0}$ on which the Brownian motion B is defined one considers the following classes of stochastic processes:

$$\mathscr{C}_0 = \{X : [0, \infty) \times \Omega \to R : X \text{ is a continuous, adapted process which}$$
$$\text{is constant at } t = 0 \text{ and } E(X^*(T)^p) < \infty,\ (\forall)\ T, p > 0\},$$

where $X^*(T) = \sup_{t \leqslant T} |X(t)|$;

$$\mathscr{C}_{k+1} = \left\{X(t) = X(0) + \sum_{j=0}^{d} \int_0^t X^i(s)\, dB^i(s) : X^i \in \mathscr{C}_k,\ 0 \leqslant i \leqslant d\right\},$$

where $B = (B^1, ..., B^d)$ is the above-mentioned Brownian motion and $B^0(t) = t$;

$$\mathscr{C}_\infty = \bigcap_{k=0}^{\infty} \mathscr{C}_k.$$

One defines $Pr_i: \mathscr{C}_1 \to \mathscr{C}_0$, $0 \leqslant i \leqslant d$, by $Pr_i(X) = X^i$, $0 \leqslant i \leqslant d$ (indistinguishable processes are identified), and, for a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$, $0 \leqslant \alpha_i \leqslant d$, one defines $Pr_\alpha: \mathscr{C}_\infty \to \mathscr{C}_\infty$ by $Pr_\alpha = Pr_{\alpha_m} \circ \cdots \circ Pr_{\alpha_1}$. One also denotes $pr_\alpha(X) = Pr_\alpha(X)(0)$ which is a real constant. For the multi-index α one puts $|\alpha| = m$ and $p(\alpha) = \#\{\alpha_i : \alpha_i \neq 0\} + 2\#\{\alpha_i : \alpha_i = 0\}$. To get unitary notation one also considers the void index $\varnothing$ and defines $Pr_\varnothing(X) = X$, $pr_\varnothing(X) = X(0)$, and $|\varnothing| = p(\varnothing) = 0$.

Then, for any process $X \in \mathscr{C}_\infty$ and any $k \in N$ one has the following Taylor type formula

$$X = \sum_{p(\alpha) \leqslant k} pr_\alpha(X)\, B^{(\alpha)} + R_k(X), \tag{b}$$

where $B^{(\alpha)}(t) = \int_0^t B_s^{(\alpha_1, ..., \alpha_{m-1})}\, dB^{\alpha_m}(s)$ and $R_k(X) \in \mathscr{C}_\infty$. One denotes

$$o(X) = \min\{p(\alpha) : pr_\alpha(X) \neq 0\}.$$

The reason for which we are interested in (b) is the following. If $c_\alpha \neq 0$ for at least one α, then, in a neighborhood of $t = 0$, $\sum_k =: \sum_{p(\alpha) = k} c_\alpha B^{(\alpha)}$ is equivalent to $\varepsilon^{k/2}$, in the sense that for every $0 < u < 1/6$

$$(S_{k/2})\ \varepsilon \to P\left(\sum_k{}^* (\varepsilon) \leqslant \varepsilon^{k/2+u}\right) \text{ is a flat function,}$$

$$(L_{k/2})\ \varepsilon \to P\left(\sum_k{}^* (\varepsilon) \geqslant \varepsilon^{k/2-u}\right) \text{ is a flat function.}$$

We recall that a function $f: R_+ \to R_+$ is called flat if $\lim_{\varepsilon \to 0} f(\varepsilon)/\varepsilon^q = 0$ for every $q \in N$. On the other hand, $R_k(X)$ fulfils $(L_{(k+1)/2})$ and thus is a "small" term. One concludes that formula (b) permits one to break X in a sum of terms of different "powers" $k/2$, $k \in N$.

We are now able to present our result. Let U be an $n \times d$ dimensional matrix with elements in $\mathscr{C}_\infty$ and

$$S_\infty(U) = \operatorname{Span}\{pr_\alpha(U_j); 0 \leqslant j \leqslant d, \alpha \text{ multi-index}\} \subseteq R^n,$$

where U_j, $1 \leqslant j \leqslant d$, are the column vectors of U. Let us also define the process

$$\left(\int U\hat{U}\right)(t) =: \int_0^t U(s)\, \hat{U}(s)\, ds.$$

The following assertions are equivalent:

(i) $\dim S_\infty(U) = n$,

(ii) $o(\det(\int U\hat{U})) < \infty$,

(iii) $\det(\int U\hat{U})$ fulfils $(S_{k/2})$ for some $k \in N$,

(iv) there is some $q \geqslant 0$ such that

$$\lim_{T\to 0} T^q E\left(\left[1/\det\left(\int U\hat{U}\right)(T)\right]^p\right)^{1/p} = 0 \qquad \text{for every} \quad p \in N.$$

Note that if the elements of U are "analytic processes" (i.e., $R_k(U_j^i) \to^k 0$) then so is $\det(\int U\hat{U})$, and consequently $o(\det(\int U\hat{U})) = \infty$ iff $\det(\int U\hat{U}) = 0$. It follows that in this case the above assertions are equivalent to (a). In any case, they imply (a). The exact distance between (iii) and (a) is emphasized in Remark 3.4 and the complete form of the result is contained in Theorem 4.1.

In order to apply the above result in the frame of the stochastic differential equations one notices that for $U_x(s) = Z_s\varphi(X(s, x, \cdot))$ one has $\mathscr{D}_{\infty, x} = S_\infty(U_x)$ and so (i) is (H_x), i.e., the Hörmander condition. The result is contained in Theorem 5.1.

Most of the authors who dealt with Malliavin calculus considered homogeneous diffusions (e.g., Malliavin [8], Ikeda and Watanabe [4], Watanabe [9], Kusuoka and Stroock [6]). Although we have not mentioned it up to now, we consider non-homogeneous diffusions and consequently the form of $\mathscr{D}_{\infty, x}$ is quite different from the one presented above (see Sect. 5). The non-homogeneous case has already been mentioned in Kosuoka and Stroock [6] who presented it as an example in the more general frame of non-Markov equations. On the other hand Ichihara and Kunita give in [3] an analytic (i.e., based on Hörmander's theorem) approach to the same problem.

1. Preliminaries

We begin by classifying the stochastic processes according to the speed of their start from $t = 0$. To this end we recall the classical notion of a flat function:

Definition 1.1. $f\colon [0, \infty) \to R$ is called flat if

$$\lim_{\varepsilon\to 0} \varepsilon^{-p} f(\varepsilon) = 0 \qquad \text{for every} \quad p \in N. \tag{1.1}$$

Clearly, f is flat iff for every $p \in N$ there is some $\varepsilon_p > 0$ such that $f(\varepsilon) \leqslant \varepsilon^p$ for every $0 < \varepsilon \leqslant \varepsilon_p$. If f is smooth, then f is flat iff all its derivatives in $x = 0$ are null. An immediate example of a flat function is $f(x) = \exp(-1/x)$ for $x > 0$ and $f(0) = 0$. We shall often use the following two elementary properties of the flat functions: if f is flat and $a, b > 0$ then $x \to f(ax^b)$ is also flat. If f and g are flat then so is $f+g$.

Our interest in flat functions comes from the following elementary remark:

Remark 1.2. Let X be a random variable on some probability space. Then

(i) $E(|x|^p) < \infty$ for every $p \in N$ iff $\varepsilon \to P(|X| \geq \varepsilon^{-1})$ is flat.

(ii) $E(|X|^{-p}) < \infty$ for every $p \in N$ iff $\varepsilon \to P(|X| \leq \varepsilon)$ is flat.

Let us now consider a probability spaces $(\Omega, \mathscr{F}, P)$ with a standard (i.e., right continuous and complete) filtration $(\mathscr{F}_t)_{t \geq 0}$ and denote

$$\begin{aligned}\mathscr{C} = \{X\colon [0, \infty) \times \Omega \to R\colon & X \text{ is a continuous adapted process,}\\ & X(0, \omega) \text{ is almost surely constant and}\\ & E(|X^*(T)|^p) < \infty \text{ for every } p \in N\},\end{aligned}$$

where $X^*(T) =: \sup_{t \leq T} |X(t)|$.

In the sequel we make the convention of identifying the indistinguishable processes (i.e., X and Y such that $X(t) = Y(t)$ for every $t \geq 0$ almost surely).

For $q > 0$ we consider the properties

(L_q) for sufficiently small $u > 0$

$$\varepsilon \to P(\sup_{t \leq \varepsilon} |X(t) - X(0)| \geq \varepsilon^{q-u}) \text{ is flat;} \tag{1.2}$$

(S_q) for sufficiently small $u > 0$

$$\varepsilon \to P(\sup_{t \leq \varepsilon} |X(t)| \leq \varepsilon^{q+u}) \text{ is flat.} \tag{1.3}$$

We denote

$$\mathscr{L}_q = \{X \in \mathscr{C}\colon X \text{ fulfils } (L_q)\}$$
$$\mathscr{S}_q = \{X \in \mathscr{C}\colon X \text{ fulfils } (S_q)\}.$$

Remark 1.3. We have the following simple examples:

(i) $X(t) = t^q \in \mathscr{L}_q \cap \mathscr{S}_q$.

(ii) Let B be a one-dimensional Brownian motion on $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t \geq 0}$. Then $B \in \mathscr{L}_{1/2} \cap \mathscr{S}_{1/2}$.

Proof. Part (i) is evident. To prove (ii) one uses the scale property to obtain $P(B^*(\varepsilon) \geq \varepsilon^{1/2-u}) = P(B^*(1) \geq \varepsilon^{-u})$.

Since $B \in \mathscr{C}$ (i.e., has finite moments of any order), Remark 1.2 ensures that the above function is flat. So $(L_{1/2})$ is proved. By again using the scale property one sees that $(S_{1/2})$ is equivalent to

$$\varepsilon \to P(B^*(1) \leqslant \varepsilon^u) \text{ is flat.}$$

Since there is no loss of generality in taking $u = 1$, we write

$$\begin{aligned} P(B^*(1) \leqslant \varepsilon) &\leqslant P(B^*(\tfrac{1}{2}) \leqslant \varepsilon, |B(1) - B(\tfrac{1}{2})| \leqslant 2\varepsilon) \\ &\leqslant P(|B(1)| \leqslant 2\varepsilon)^2 = \frac{1}{\pi}\left(\int_{\{|x| \leqslant 2\varepsilon\}} e^{-x^2}\, dx\right)^2 \leqslant \frac{16}{\pi}\varepsilon^2. \end{aligned}$$

The same procedure permits one to prove that for every $p \in N$ there is some constant K_p such that $P(B^*(1) \leqslant \varepsilon) \leqslant K_p \varepsilon^p$ which completes the proof of $(S_{1/2})$. Q.E.D.

We list now some elementary properties of $\mathscr{L}_q$ and $\mathscr{S}_q$:

Proposition 1.4. (A)(i) *$\mathscr{L}_{q'} \subseteq \mathscr{L}_q$ for $q \leqslant q'$; $\mathscr{L}_0 = \mathscr{C}$.*

(ii) *If $X, Y \in \mathscr{L}_q$ then $X + Y \in \mathscr{L}_q$.*

(iii) *If $X \in \mathscr{L}_q$, $Y \in \mathscr{L}_{q'}$, and $X(0) = Y(0) = 0$ a.s. then $XY \in \mathscr{L}_{q+q'}$. In particular, $ZX \in \mathscr{L}_q$ if $Z \in \mathscr{C} = \mathscr{L}_0$.*

(iv) *If B is a one-dimensional Brownian motion and $X \in \mathscr{L}_q$ then $t \to \int_0^t (X_s - X_0)\, dB_s$ is in $\mathscr{L}_{q+1/2}$ and $t \to \int_0^t (X_s - X_0)\, ds$ is in $\mathscr{L}_{q+1}$.*

(B) (i) *$\mathscr{S}_q \subseteq \mathscr{S}_{q'}$ for $q \leqslant q'$.*

(ii) *If $X \in \bigcup_{0<q<\infty} \mathscr{S}_q$ then $E(X^*(T)^{-p}) < \infty$ for every $p, T > 0$.*

(C) *If $X \in \mathscr{S}_q$ and $Y \in \mathscr{L}_{q'}$ for some $q < q'$ then $X + Y \in \mathscr{S}_q$.*

Proof. Parts (A)(i), (ii), and (iii) are straightforward. To prove (iv) we note that $m(t) = \int_0^t (X_s - X_0)\, dB_s$ is a martingale whose increasing process is $\langle m \rangle(t) = \int_0^t (X_s - X_0)^2\, ds$. Then there is some one-dimensional Brownian motion (see, e.g., Ikeda and Watanabe [4]) b such that $m = b \circ \langle m \rangle$. One writes

$$\begin{aligned} P(m^*(\varepsilon) \geqslant \varepsilon^{q+1/2-u}) &= P(b^*(\langle m \rangle(\varepsilon)) \geqslant \varepsilon^{q+1/2-u}) \\ &\leqslant P(\langle m \rangle(\varepsilon) \geqslant \varepsilon^{2q+1-u}) + P(b^*(\langle m \rangle(\varepsilon)) \\ &\geqslant \varepsilon^{q+1/2-u}, \langle m \rangle(\varepsilon) \leqslant \varepsilon^{2q+1-u}) \\ &=: f(\varepsilon) + f'(\varepsilon). \end{aligned}$$

One has

$$f(\varepsilon) \leqslant P(\sup_{t \leqslant \varepsilon} |X_t - X_0|^2 \geqslant \varepsilon^{2q-u}) = P(\sup_{t \leqslant \varepsilon} |X_t - X_0| \geqslant \varepsilon^{q-u/2}),$$

which, in view of (L_q) for X, is a flat function.

Next

$$f'(\varepsilon) \leqslant P(b^*(\varepsilon^{2q+1-u}) \geqslant \varepsilon^{q+1/2-u})$$

which, in view of $(L_{1/2})$ for b, is also flat.

The other assertions in Proposition 1.4 are elementary so we omit their proof. Q.E.D.

Let us now define the "global" analogs of properties (L_q) and (S_q). For $f\colon [0, \infty) \to R$ and $T, \varepsilon > 0$ one denotes

$$\delta_{T,\varepsilon}(f) = \sup\{|f(t) - f(T)| : T \leqslant t \leqslant T + \varepsilon\}$$

and considers the properties:

$(\hat{L}_{1/2})$ For every $T, u > 0$ and every family of stopping times $0 \leqslant T_\varepsilon \leqslant T$, $\varepsilon > 0$,

$$\varepsilon \to P(\delta_{T_\varepsilon,\varepsilon}(X) \geqslant \varepsilon^{1/2-u}) \text{ is flat}$$

and

$(\hat{S})$ For every $T > 0$
$\varepsilon \to P(X^*(T) \leqslant \varepsilon)$ is flat.

Clearly, $(\hat{L}_{1/2})$ is a stronger version of $(L_{1/2})$. To emphasize the relation between $(\hat{S})$ and (S_q) we have to consider a process X which has a "q-scale property," i.e., the process $(X(t))_{t \geqslant 0}$ has the same distribution as $(a^q X(t/a))_{t \geqslant 0}$ for every $a > 0$. Such processes appear when considering multiple stochastic integrals (see (2.1)). For such a process one has

$$P(\sup_{t \leqslant \varepsilon} |X(t)| \leqslant \varepsilon^{q+u}) = P(\sup_{t \leqslant 1} |X(t)| \leqslant \varepsilon^u)$$

and so $(\hat{S})$ is equivalent to (S_q).

PROPOSITION 1.5. *Let* $B = (B^1, ..., B^d)$ *be a Brownian motion,* $X_i \in \mathscr{C}$, $0 \leqslant i \leqslant d$, *and*

$$X(t) = \sum_{i=1}^{d} \int_0^t X_i(s)\, dB^i(s) + \int_0^t X_0(s)\, ds.$$

(i) *Let* $Y = (\sum_{i=1}^d X_i^2)^{1/2}$. *If* Y *fulfils* $(\hat{L}_{1/2})$ *and* $(\hat{S})$ *then* X *fulfils* $(\hat{S})$.

(ii) *If* $Y = 0$, X_0 *fulfils* $(\hat{L}_{1/2})$ *and* $(\hat{S})$ *then* X *fulfils* $(\hat{S})$.

Both properties $(\hat{S})$ and $(\hat{L}_{1/2})$ and the above proposition will be considered in a more complex frame (see Proposition 2.3) so we do not give the proof of Proposition 1.5 here.

2. Multiple Integrals

Let $B=(B^1, \dots, B^d)$ be a Brownian motion on $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t\geqslant 0}$. To get unitary notation we put $B^0(t)=t$. We shall define multiple integrals with respect to B^i, $0\leqslant i\leqslant d$, the integral with respect to B^0 being the usual Lebesgue integral and the one with respect to B^i, $1\leqslant i\leqslant d$, the Itô integral.

For a multi-index $\alpha=(\alpha_1, \dots, \alpha_n)$, $0\leqslant \alpha_i\leqslant d$, $1\leqslant i\leqslant n$, we shall denote

$$\begin{aligned}|\alpha| &= n\\ p(\alpha) &= 2\#\{1\leqslant i\leqslant n:\alpha_i=0\}+\#\{1\leqslant i\leqslant n:\alpha_i\neq 0\}\\ &= |\alpha|+\#\{1\leqslant i\leqslant n:\alpha_i=0\}.\end{aligned}$$

We shall consider also the multi-index $\alpha=\varnothing$. In this case $|\varnothing|=p(\varnothing)=0$. For $n\geqslant 1$ one denotes

$$\underline{\alpha}=(\alpha_1, \dots, \alpha_{n-1}) \qquad \text{and} \qquad \bar{\alpha}=\alpha_n.$$

For a process $X\in\mathscr{C}$ and a multi-index α we define the multiple stochastic integral

$$\begin{aligned} I^{(\alpha)}(X, B)(t) &= X(t) && \text{for } \alpha=\varnothing\\ I^{(\alpha)}(X, B)(t) &= \int_0^t I^{(\underline{\alpha})}(X, B)(s)\, dB^{\bar{\alpha}}(s) && \text{for } |\alpha|\geqslant 1.\end{aligned}$$

In the case $X=1$ one puts $B^{(\alpha)}=I^{(\alpha)}(1, B)$, i.e.,

$$\begin{aligned} B^{(\alpha)}(t) &= 1 && \text{for } \alpha=\varnothing\\ B^{(\alpha)}(t) &= \int_0^t B^{(\underline{\alpha})}(s)\, dB^{\bar{\alpha}}(s) && \text{if } |\alpha|\geqslant 1.\end{aligned}$$

Let us note that the multiple integrals $B^{(\alpha)}$ inherit from B the scale property. More exactly, for $a>0$ let

$$B_a^i(t/a)=a^{1/2}B^i(t/a),\ 1\leqslant i\leqslant d, \qquad \text{and} \qquad B_a^0(t)=t.$$

Then, for almost all $\omega\in\Omega$ one has

$$B_a^{(\alpha)}(t)=a^{p(\alpha)/2}B^{(\alpha)}(t/a), \tag{2.1}$$

and consequently the processes $(B^{(\alpha)}(t))_{t\geqslant 0}$ and $(a^{p(\alpha)/2}B^{(\alpha)}(t/a))_{t\geqslant 0}$ are identically distributed.

Let us also note the following elementary properties of the multiple stochastic integrals:

$$\begin{aligned} &\text{(i)} \quad \text{If } X \in \mathscr{L}_q \text{ then } I^{(\alpha)}(X, B) \in \mathscr{L}_{q+p(\alpha)/2}. \\ &\text{(ii)} \quad \text{If } X \in \mathscr{C} \text{ then } I^{(\alpha)}(X, B) \text{ fulfils } (\hat{L}_{1/2}). \end{aligned} \tag{2.2}$$

To check (i) one applies Proposition 1.4(A)(iv) inductively and to prove (ii) one uses the time change procedure already presented in the proof of Proposition 1.4(A)(iv) and property $(\hat{L}_{1/2})$ for the Brownian motion.

The following proposition contains the key idea in this section: the stochastic multiple integrals $B^{(\alpha)}$ with the same "power" $p(\alpha) = k$ act in an independent way, i.e., do not annihilate each other. It follows that any linear combination of such integrals yields a process of the same power, i.e., which has the properties $(S_{k/2})$ and $(L_{k/2})$. This allows us to decompose a process X—by means of Taylor series—in a sum of processes of powers $k/2$, $k = 0, 1, 2, \ldots$. This will be done in Section 3. On the other hand the idea in Proposition 2.1 will be lifted at a higher level in Proposition 2.2 in which we consider linear combinations with certain stochastic processes as coefficients (instead of constants).

PROPOSITION 2.1. *Let* $X = \sum_{p(\alpha)=k} c_\alpha B^{(\alpha)}$, $c_\alpha \in R$. *The following assertions are equivalent*:

(i) $X \in \mathscr{S}_{k/2}$.

(ii) *X fulfils* $(\hat{S})$.

(iii) $X \neq 0$ (*indistinguishable processes are identified*).

(iv) $c_\alpha \neq 0$ *for at least one*.

Proof. The equivalence between (i) and (ii) is an immediate consequence of the scale property (2.1) and the equivalence between (iii) and (iv) is obtained by using inductively the classical assertion $\sum_{i=1}^{d} Z_i dB^i + Z_0\, dt = 0$ iff $Z_i = 0$, $0 \leqslant i \leqslant d$. The implication (i) $\Rightarrow$ (iii) is obvious and (iv) $\Rightarrow$ (ii) is obtained by using Proposition 1.5 inductively.

Q.E.D.

We shall now state the version of Proposition 2.1 we announced above. Let d and d' be two non-negative integers let $B = (B^1, \ldots, B^d)$ and $\hat{B} = (\hat{B}^1, \ldots, \hat{B}^{d'})$ be two independent Brownian motions on $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t \geqslant 0}$, and let us denote $\mathscr{B}_t = \sigma(B_s, s \leqslant t)$ and $\hat{\beta}_t = \sigma(\hat{B}_s, s \leqslant t)$. For some $n, m \in N$ we define

$$\lambda_\alpha = \sum_{p(\alpha)=m} c_{\alpha,\beta}\, \hat{B}^{(\beta)}, \qquad \text{with} \quad c_{\alpha,\beta} \in R,$$

$$Z_{n,m}(s, t) = \sum_{p(\beta)=n} \lambda_\alpha(t)\, B^{(\alpha)}(s).$$

Here α is a multi-index with components in $\{0, 1, ..., d\}$ and β is a multi-index with components in $\{0, 1, ..., d'\}$.

PROPOSITION 2.2. *Assume that there is some* $c_{\alpha,\beta} \neq 0$. *Then, for every* $T, \eta > 0$ *there exists a family of stopping times* (*with respect to* $\hat{\beta}_t$, $t \geqslant 0$), $0 \leqslant T_\varepsilon^\eta \leqslant T$, $\varepsilon > 0$, *such that*

$$\varepsilon \to P(\sup_{s \leqslant 1} |Z_{n,m}(s, T_\varepsilon^\eta)| \leqslant \varepsilon^\eta) \textit{ is flat.} \tag{2.3}$$

In order to prove the above proposition we have to state the result in Proposition 1.5 in a more sophisticated frame, that of families of stochastic processes. Let $X_{\varepsilon,\eta} \in \mathscr{C}$, $\varepsilon, \eta > 0$, and $a > 0$. We need the following versions of $(\hat{S})$ and $(\hat{L}_{1/2})$:

$(\hat{S}_a)$ For every $T, \eta > 0$

$$\varepsilon \to P(X_{\varepsilon,\eta}^*(T) \leqslant \varepsilon^{a\eta}) \text{ is flat.}$$

Clearly, if $X_{\varepsilon,\eta}$ does not depend on $\varepsilon, \eta > 0$ then $(\hat{S}_a)$ is exactly $(\hat{S})$.

$(\hat{L}_{1/2})$ For every $T, \gamma, \eta, u > 0$ and every family of stopping times $0 \leqslant t_{\varepsilon,\eta} \leqslant T$, $\varepsilon > 0$,

$$\varepsilon \to P(\delta_{t_{\varepsilon,\eta},\varepsilon^\gamma}(X_{\varepsilon,\eta}) \geqslant \varepsilon^{1/2 - u}) \text{ is flat.}$$

If $X_{\varepsilon,\eta}$ does not depend on $\varepsilon, \eta > 0$ then $(\hat{L}_{1/2})$ above is exactly that stated in Section 1.

$(\hat{I})$ $\sup_{\varepsilon,\eta} E(X_{\varepsilon,\eta}^*(T)^p) < \infty$ for every $T, p > 0$.

If $X_{\varepsilon,\eta} = X \in \mathscr{C}$ for every $\varepsilon, \eta > 0$ then $(\hat{I})$ is clearly fulfiled.
We need the following version of Proposition 1.5.

PROPOSITION 2.3. *Let* $X_{\varepsilon,\eta}^i \in \mathscr{C}$, $\varepsilon, \eta > 0$, $0 \leqslant i \leqslant d$, *and*

$$X_{\varepsilon,\eta}(t) = \sum_{i=1}^d \int_0^t X_{\varepsilon,\eta}^i(s)\, dB^i(s) + \int_0^t X_{\varepsilon,\eta}^0(s)\, ds.$$

(i) *If* $Y_{\varepsilon,\eta} =: (\sum_{i=1}^d (X_{\varepsilon,\eta}^i)^2)^{1/2}$, $\varepsilon, \eta > 0$, *fulfils* $(\hat{S}_{a/3})$, $(\hat{L}_{1/2})$, *and* $(\hat{I})$ *and* $X_{\varepsilon,\eta}^0$, $\varepsilon, \eta > 0$, *fulfils* $(\hat{I})$ *then* $X_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$, *fulfils* $(\hat{S}_a)$.

(ii) *If* $Y_{\varepsilon,\eta} = 0$, $\varepsilon, \eta > 0$, *and* $X_{\varepsilon,\eta}^0$, $\varepsilon, \eta > 0$, *fulfils* $(\hat{S}_{a/8})$, $(\hat{L}_{1/2})$, *and* $(\hat{I})$ *then* $X_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$, *fulfils* $(\hat{S}_a)$.

Proof. Let us fix $T, a, \eta > 0$ and define

$$h_{\varepsilon,\eta} = \varepsilon^{(11/30)a\eta}, \qquad l_{\varepsilon,\eta} = h_{\varepsilon,\eta}^3 = \varepsilon^{(11/10)a\eta}$$

and

$$\begin{aligned}
t_{\varepsilon,\eta} &= \inf\{0 \leqslant t \leqslant T/2 \colon |Y_{\varepsilon,\eta}(t)| \geqslant 2h_{\varepsilon,\eta}\} \\
&= \infty \qquad \text{if} \quad \{\cdots\} = \emptyset, \\
t'_{\varepsilon,\eta} &= \inf\{t_{\varepsilon,\eta} \leqslant t \leqslant T \colon |Y_{\varepsilon,\eta}(t)| > h_{\varepsilon,\eta}\} \\
&= \infty \qquad \text{if} \quad \{\cdots\} = \emptyset.
\end{aligned}$$

One writes

$$X_{\varepsilon,\eta} = m_{\varepsilon,\eta} + n_{\varepsilon,\eta}$$

with

$$m_{\varepsilon,\eta} = \sum_{i=1}^{d} X^i_{\varepsilon,\eta}\, dB^i \qquad \text{and} \qquad n_{\varepsilon,\eta} = X^0_{\varepsilon,\eta}\, dt.$$

Note that

$$\begin{aligned}
&\{t_{\varepsilon,\eta} \leqslant T/2,\ X^*_{\varepsilon,\eta}(T) \leqslant \varepsilon^{a\eta}\} \\
&\quad \subseteq \{t_{\varepsilon,\eta} \leqslant T/2,\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(m_{\varepsilon,\eta}) \leqslant 4\varepsilon^{a\eta}\} \\
&\quad \cup \{t_{\varepsilon,\eta} \leqslant T/2,\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(n_{\varepsilon,\eta}) \geqslant \varepsilon^{a\eta}\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
P(X^*_{\varepsilon,\eta}(T) \leqslant \varepsilon^{a\eta}) \leqslant P(t_{\varepsilon,\eta} = \infty) \\
&+ P(t_{\varepsilon,\eta} \leqslant T/2,\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(m_{\varepsilon,\eta}) \leqslant 4\varepsilon^{a\eta}) \\
&+ P(t_{\varepsilon,\eta} \leqslant T/2,\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(n_{\varepsilon,\eta}) \geqslant \varepsilon^{a\eta}) \\
&=: f^{(1)}_{\varepsilon,\eta} + f^{(2)}_{\varepsilon,\eta} + f^{(3)}_{\varepsilon,\eta}.
\end{aligned}$$

Since

$$f^{(1)}_{\varepsilon,\eta} = P(Y^*_{\varepsilon,\eta}(T/2) \leqslant 2^{(11/30)a\eta})$$

and $Y_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$, fulfils $(\hat{S}_{a/3})$, it follows that $\varepsilon \to f^{(1)}_{\varepsilon,\eta}$ is flat.

To deal with $f^{(3)}_{\varepsilon,\eta}$ one writes

$$\begin{aligned}
f^{(3)}_{\varepsilon,\eta} \leqslant P((X^0_{\varepsilon,\eta})^*\,(T)\, l_{\varepsilon,\eta} \geqslant \varepsilon^{a\eta}) &= P((X^0_{\varepsilon,\eta})^*\,(T) \geqslant \varepsilon^{-a\eta/10}) \\
&\leqslant \varepsilon^{pa\eta/10} E((X^0_{\varepsilon,\eta})^*\,(T)^p) \\
&\geqslant \varepsilon^{pa\eta/10} \sup_{\varepsilon,\eta>0} E((X^0_{\varepsilon,\eta})^*\,(T)^p).
\end{aligned}$$

Since the above inequality holds for every $p \in N$ and the family $X^0_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$, fulfils $(\hat{I})$, one concludes that $\varepsilon \to f^{(3)}_{\varepsilon,\eta}$ is flat.

Let us now write

$$\begin{aligned} f^{(2)}_{\varepsilon,\eta} &\leqslant P(t_{\varepsilon,\eta} \leqslant T/2,\ t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} \leqslant l_{\varepsilon,\eta}) \\ &\quad + P(t_{\varepsilon,\eta} \leqslant T/2,\ t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} > l_{\varepsilon,\eta},\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(m_{\varepsilon,\eta}) \leqslant 4\varepsilon^{a\eta}) \\ &=: f^{(4)}_{\varepsilon,\eta} + f^{(5)}_{\varepsilon,\eta}. \end{aligned}$$

We take $\gamma = 10/(11a\eta)$ and write

$$\begin{aligned} f^{(4)}_{\varepsilon,\eta} &\leqslant P(t_{\varepsilon,\eta} \leqslant T/2,\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(Y_{\varepsilon,\eta}) \geqslant h_{\varepsilon,\eta}) \\ &= P(t_{\varepsilon,\eta} \leqslant T/2,\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(Y_{l^{\gamma}_{\varepsilon,\eta},\eta}) \geqslant l^{1/3}_{\varepsilon,\eta}), \end{aligned}$$

which, by $(\hat{L}_{1/2})$ for $Y_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$ (one takes $\bar{\varepsilon} = l_{\varepsilon,\eta}$ instead of ε), is flat.

Let us now evaluate $f^{(5)}_{\varepsilon,\eta}$. Let $b_{\varepsilon,\eta}$ be a Brownian motion such that $m_{\varepsilon,\eta} = b_{\varepsilon,\eta} \circ \langle m_{\varepsilon,\eta} \rangle$. Note that $s_{\varepsilon,\eta} = \langle m_{\varepsilon,\eta} \rangle (t_{\varepsilon,\eta})$ is a stopping time with respect to the filtration associated with $b_{\varepsilon,\eta}$ and, if $t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} \geqslant l$, then

$$\begin{aligned} &\langle m_{\varepsilon,\eta} \rangle (t_{\varepsilon,\eta} + l_{\varepsilon,\eta}) - \langle m_{\varepsilon,\eta} \rangle (t_{\varepsilon,\eta}) \\ &\quad = \int_{t_{\varepsilon,\eta}}^{t_{\varepsilon,\eta} + l_{\varepsilon,\eta}} Y^2_{\varepsilon,\eta}(s)\, ds \geqslant h^2_{\varepsilon,\eta} \cdot l_{\varepsilon,\eta} = \varepsilon^{(11/6)a\eta}. \end{aligned}$$

It follows that

$$f^{(5)}_{\varepsilon,\eta} \leqslant P(\delta_{s_{\varepsilon,\eta}, \varepsilon^{11a\eta/6}}(b_{\varepsilon,\eta}) \leqslant 4\varepsilon^{a\eta}) = P(b^*(\varepsilon^{11a\eta/6}) \leqslant 4\varepsilon^{a\eta}),$$

where b designates some anonymous Brownian motion. In view of $(L_{1/2})$ for b, $\varepsilon \to f^{(5)}_{\varepsilon,\eta}$ is flat and so the proof of (i) is completed.

Let us now prove (ii). One takes

$$h_{\varepsilon,\eta} = \varepsilon^{a\eta/8}, \qquad l_{\varepsilon,\eta} = h^3_{\varepsilon,\eta} = \varepsilon^{3a\eta/8},$$

and defines $t_{\varepsilon,\eta}$ and $t'_{\varepsilon,\eta}$ as above but with $X^0_{\varepsilon,\eta}$ instead of $Y_{\varepsilon,\eta}$. Then

$$\begin{aligned} &P(X^*_{\varepsilon,\eta}(T) \leqslant \varepsilon^{a\eta}) \\ &\quad = P(n^*_{\varepsilon,\eta}(T) \leqslant \varepsilon^{a\eta}) \leqslant P(t_{\varepsilon,\eta} = \infty) \\ &\qquad + P(t_{\varepsilon,\eta} \leqslant T/2),\ t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} \leqslant l_{\varepsilon,\eta}) \\ &\qquad + P(t_{\varepsilon,\eta} \leqslant T/2,\ t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} > l_{\varepsilon,\eta},\ \delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(n_{\varepsilon,\eta}) \leqslant 3\varepsilon^{a\eta}) \\ &\quad =: f^{(6)}_{\varepsilon,\eta} + f^{(7)}_{\varepsilon,\eta} + f^{(8)}_{\varepsilon,\eta}. \end{aligned}$$

To prove that $\varepsilon \to f^{(6)}_{\varepsilon,\eta}$ respectively $\varepsilon \to f^{(7)}_{\varepsilon,\eta}$ are flat, one uses the same argument as that used above for $\varepsilon \to f^{(1)}_{\varepsilon,\eta}$ respectively $\varepsilon \to f^{(4)}_{\varepsilon,\eta}$.

To dominate $f^{(8)}_{\varepsilon,\eta}$ one notes that $t \to X^0_{\varepsilon,\eta}(t)$ does not change its sign on $[t_{\varepsilon,\eta}, t_{\varepsilon,\eta} + l_{\varepsilon,\eta}]$ and consequently

$$\delta_{t_{\varepsilon,\eta}, l_{\varepsilon,\eta}}(n_{\varepsilon,\eta}) = \int_{t_{\varepsilon,\eta}}^{t_{\varepsilon,\eta}+l_{\varepsilon,\eta}} |X^0_{\varepsilon,\eta}(s)|\, ds \geqslant h_{\varepsilon,\eta} \cdot l_{\varepsilon,\eta} = \varepsilon^{a\eta/2}.$$

It follows that for sufficiently small ε

$$f^3_{\varepsilon,\eta} \leqslant P(\varepsilon^{a\eta/2} \geqslant 3\varepsilon^{\eta a}) = 0$$

and so the proof of (ii) is also completed. Q.E.D.

Proof of Proposition 2.2. We shall proceed by induction over n. For $n = 0$ the only multi-index α such that $p(\alpha) = 0$ is $\alpha = \varnothing$. Since $B^{(\varnothing)}(s) = 1$, (2.3) reduces to

$$\varepsilon \to P(|\lambda_\varnothing(T^\eta_\varepsilon)| \leqslant \varepsilon^\eta) \text{ flat.}$$

Let us take

$$T^\eta_\varepsilon = \inf\{0 < t \leqslant T\colon |\lambda_\varnothing(t)| > \varepsilon^\eta\} = \infty \qquad \text{if} \quad \{\cdots\} = \varnothing.$$

Clearly T^η_ε is a stopping time with respect to $(\hat{\beta}_t)_{t \geqslant 0}$ and

$$P(|\lambda_\varnothing(T^\eta_\varepsilon)| \leqslant \varepsilon^\eta) = P(T^\eta_\varepsilon = \infty) = P(\lambda^*_\varnothing(T) \leqslant \varepsilon^\eta),$$

which, by Proposition 2.1, is flat.

Let us now assume that (2.3) is proved for n and check it for $n+1$. First assume that there is some α_0 with $i_0 = \bar{\alpha}_0 \in \{1, ..., d\}$ and β_0 such that $c_{\alpha_0, \beta_0} \neq 0$. Consider some $\eta > 0$. Then, in view of the induction hypothesis one may find some stopping times (with respect to $(\hat{\beta}_t)_{t \geqslant 0}$), $0 \leqslant T^{\eta/3}_\varepsilon \leqslant T$, $\varepsilon > 0$, such that

$$\varepsilon \to P\left(\sup_{s \leqslant 1} \left| \sum_{\substack{p(\alpha) = n+1 \\ \bar{\alpha} = i_0}} \lambda_\alpha(T^{\eta/3}_\varepsilon)\, B^{(\alpha)}(s) \right| \leqslant \varepsilon^{\eta/3}\right) \text{ is flat.} \tag{2.4}$$

Let us prove that this is the family of stopping times we are looking for. Denote

$$Y^i_{\varepsilon,\eta}(s) = \sum_{\substack{p(\alpha) = n+1 \\ \bar{\alpha} = i}} \lambda_\alpha(T^{\eta/3}_\varepsilon)\, B^{(\alpha)}(s), \qquad 0 \leqslant i \leqslant n,$$

and

$$Y_{\varepsilon,\eta} = \left(\sum_{i=1}^d (Y^i_{\varepsilon,\eta})^2\right)^{1/2}.$$

In view of (2.4), the family $Y^{i_0}_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$, fulfils $(\hat{S}_{1/3})$ and consequently $Y_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$, fulfils the same property. Let us check that it fulfils $(\hat{L}_{1/2})$. To this end it will suffice to check this property for each of $s \to \lambda_\alpha(T^{\eta/3}_\varepsilon)\, B^{(\alpha)}(s)$, $\varepsilon, \eta > 0$. We fix $T, \eta, \gamma, u > 0$ and $0 \leqslant t_{\varepsilon,\eta} \leqslant T$ and write

$$P(\delta_{t_{\varepsilon,\eta},\varepsilon}(\lambda_\alpha(T^{\eta/3}_\varepsilon)\, B^{(\alpha)}(\cdot)) \geqslant \varepsilon^{1/2-u}) \leqslant P(\lambda^*_\alpha(T) \geqslant \varepsilon^{-u/2})$$
$$+ P(\delta_{t_{\varepsilon,\eta},\varepsilon}(B^{(\alpha)}) \geqslant \varepsilon^{1/2-u/2}) =: f^{(1)}_{\varepsilon,\eta} + f^{(2)}_{\varepsilon,\eta}.$$

Since $E(\lambda^*_\alpha(T)^p) < \infty$ for every $p \in N$, $\varepsilon \to f^{(1)}_{\varepsilon,\eta}$ is flat. Next, since $B^{(\alpha)}$ fulfils $(L_{1/2})$, $\varepsilon \to f^{(2)}_{\varepsilon,\eta}$ is also flat. So $(\hat{L}_{1/2})$ is proved for $Y_{\varepsilon,\eta}$, $\varepsilon, \eta > 0$. Since property $(\hat{I})$ is obviously fulfilled we may apply Proposition 2.3(i) to conclude that the family $Z_{\varepsilon,\eta}(s) =: Z_{n,m}(s, T^{\eta/3}_\varepsilon)$, $\varepsilon, \eta > 0$, fulfils $(\hat{S}_1)$. So (2.3) is proved in this case.

Assume now that $c_{\alpha,\beta} = 0$ for every α for which $1 \leqslant \bar{\alpha} \leqslant d$. Then $c_{\alpha,\beta} \neq 0$ for some α with $\bar{\alpha} = 0$. In this case the reasoning is exactly the same as above with the only difference being that one uses Proposition 2.3(ii) instead of (i). So the proof is completed. Q.E.D.

The following proposition represents the key step in the attempt of evaluating the determinant of the Malliavin covariance matrix. In this sense it represents our alternative to Theorem (A.6) in Kusuoka and Stroock [6]. Let

$$S_k(s, t) = \sum_{p(\alpha) + p(\beta) = k} c_{\alpha,\beta} B^{(\alpha)}(s)\, B^{(\beta)}(t), \qquad c_{\alpha,\beta} \in R,$$

where $B = (B^1, ..., B^d)$ is a Brownian motion on $(\Omega, \mathscr{F}, P)$, $(\mathscr{F}_t)_{t \geqslant 0}$.

PROPOSITION 2.4. *If there is some $c_{\alpha,\beta} \neq 0$, then, for every $T > 0$,*

$$\varepsilon \to P\left(\sup_{t \leqslant T} \int_0^t S_k^2(s, t)\, ds \leqslant \varepsilon\right) \textit{ is flat.} \tag{2.5}$$

We shall reduce the above assertion to Proposition 2.2 by using the following two lemmas. Let

$$S_{n,m}(s, t) = \sum_{p(\alpha) = n} \sum_{p(\beta) = m} c_{\alpha,\beta} B^{(\alpha)}(s)\, B^{(\beta)}(t), \qquad c_{\alpha,\beta} \in R.$$

LEMMA 2.5. *If there is some $c_{\alpha,\beta} \neq 0$ then for every $T, \eta > 0$*

$$\varepsilon \to P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon S^2_{n,m}(s, t)\, ds \leqslant \varepsilon^{n+1+\eta}\right) \textit{ is flat.} \tag{2.6}$$

Proof of Proposition 2.4. (Under the assumption that Lemma 2.5 is true.) Let $n_0 = \min\{n\text{:there are some } \alpha, \beta \text{ such that } p(\alpha) = n, p(\alpha) + p(\beta) = k \text{ and } c_{\alpha,\beta} \neq 0\}$.

One writes

$$S_k = \sum_{n=n_0}^{k} S_{n,k-n}.$$

We shall prove that

$$\varepsilon \to P\left(\sup_{t \leqslant T} \int_0^t S_k^2(s, t)\, ds \leqslant \varepsilon^{n_0+1+\eta}\right) \text{ is flat,}$$

which clearly is equivalent to (2.5). Let us write

$$\begin{aligned} P&\left(\sup_{t \leqslant T} \int_0^t S_k^2(s, t)\, ds \leqslant \varepsilon^{n_0+1+\eta}\right) \\ &\leqslant P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon S_k^2(s, t)\, ds \leqslant \varepsilon^{n_0+1+\eta}\right) \\ &\leqslant P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon S_{n_0,k-n_0}^2(s, t)\, ds \leqslant 2\varepsilon^{n_0+1+\eta}\right) \\ &\quad + P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon 2\,|S_{n_0,k-n_0}(s, t)| \cdot \left|\sum_{n=n_0+1}^{k} S_{n,k-n}(s, t)\right| ds \geqslant \varepsilon^{n_0+1+\eta}\right) \\ &=: f_\varepsilon^{(1)} + f_\varepsilon^{(2)}. \end{aligned}$$

By (2.6), $\varepsilon \to f_\varepsilon^{(1)}$ is flat. Let us next notice that $S_{n_0,k-n_0}(s, t) \cdot \sum_{n=n_0+1}^{k} S_{n,k-n}(s, t)$ is a sum of terms of the form $cB^{(\alpha)}(s)\, B^{(\beta)}(t)\, B^{(\alpha')}(s)\, B^{(\beta')}(t)$ where $c \in R$, $p(\alpha) = n_0$, $p(\beta) = k - n_0$, $p(\alpha') = n \geqslant n_0 + 1$, and $p(\beta') = k - n$. Then, in order to check that $\varepsilon \to f_\varepsilon^{(2)}$ is flat, it will suffice to prove that

$$\varepsilon \to P([B^{(\alpha)}\, B^{(\alpha')}]^*\, (\varepsilon) \cdot [B^{(\beta)}\, B^{(\beta')}]^*\, (T) \geqslant \varepsilon^{n_0+\eta}) \text{ is flat.}$$

One dominates the above term by

$$\begin{aligned} &P([B^{(\alpha)}\, B^{(\alpha')}]^*\, (\varepsilon) \geqslant \varepsilon^{n_0+2\eta}) \\ &\qquad + P([B^{(\beta)}\, B^{(\beta')}]^*\, (T) \geqslant \varepsilon^{-\eta}) =: f_\varepsilon^{(3)} + f_\varepsilon^{(4)}. \end{aligned}$$

Since $B^{(\alpha)} \in \mathscr{L}_{n_0/2}$ and $B^{(\alpha')} \in \mathscr{L}_{(n_0+1)/2}$, $B^{(\alpha)}\, B^{(\alpha')} \in \mathscr{L}_{n_0+1/2}$ and so, for $\eta < \frac{1}{4}$, $\varepsilon \to f_\varepsilon^{(3)}$ is flat. On the other hand, since $E([B^{(\beta)}\, B^{(\beta')}]^*\, (T)^p) < \infty$ for every $p \in N$, $\varepsilon \to f_\varepsilon^{(4)}$ is also flat. So the proof is completed. Q.E.D.

To prove Lemma 2.5 we need some more notation. For $k \in N$ and $\varepsilon > 0$ one denotes

$$\Gamma_k = \{(u_1, ..., u_k) : 0 \leqslant u_k \leqslant \cdots \leqslant u_1\} \subseteq R^k_+,$$
$$\Gamma_k = \{(u_1, ..., u_k) \in \Gamma_k ; 0 < u_k \leqslant \varepsilon\},$$
$$\Lambda^\varepsilon_k = \{(u_1, ..., u_k) \in \Gamma_k : \varepsilon < u_k\}.$$

Clearly

$$B^{(\alpha)} = I^{(\alpha)}(\Gamma_{|\alpha|}, B) = I^{(\alpha)}(\Lambda^\varepsilon_{|\alpha|}, B) + I^{(\alpha)}(\Gamma^\varepsilon_{|\alpha|}, B),$$

where, for a measurable set $H \subseteq R^{|\alpha|}$, $I^{(\alpha)}(H, B)$ designates the multiple integral made on the set H.

Let us denote

$$S^\varepsilon_{n,m}(s, t) = \sum_{p(\alpha) = n} \sum_{p(\beta) = m} c_{\alpha,\beta} B^{(\alpha)}(s)\, I^{(\beta)}(\Lambda^\varepsilon_\beta, B)(t),$$
$$L^\varepsilon_{n,m}(s, t) = \sum_{p(\alpha) = n} \sum_{p(\beta) = m} c_{\alpha,\beta} B^\alpha(s)\, I^\beta(\Gamma^\varepsilon_\beta, B)(t).$$

We shall reduce Lemma 2.5 to

LEMMA 2.6. *If there is some $c_{\alpha,\beta} \neq 0$ then for every T, $\eta > 0$*

$$\varepsilon \to P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon (S_{n,m}(s, t))^2\, ds \leqslant \varepsilon^{n+1+\eta}\right) \textit{ is flat.} \tag{2.7}$$

Proof of Lemma 2.5. (Under the assumption that Lemma 2.6 is true.) Since $S_{n,m} = S^\varepsilon_{n,m} + L^\varepsilon_{n,m}$, if (2.7) holds, then (2.6) reduces to

$$\varepsilon \to P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon 2\, |S^\varepsilon_{n,m}(s, t)\, L^\varepsilon_{n,m}(s, t)|\, ds \geqslant \varepsilon^{n+1+\eta}\right) \text{ is flat.}$$

Let us note that $S^\varepsilon_{n,m}(s, t)\, L^\varepsilon_{n,m}(s, t)$ is a sum of terms of the form $cB^{(\alpha)}(s)\, B^{(\alpha')}(s)\, I^{(\beta)}(\Lambda^\varepsilon_{|\beta|}, B)(t)\, I^{(\beta')}(\Gamma^\varepsilon_{|\beta|}, B)(t)$, where $c \in R$, $p(\alpha) = p(\alpha') = n$, and $p(\beta) = p(\beta') = m$. So, what is to be proved is

$$\varepsilon \to P([B^{(\alpha)} B^{(\alpha')}]^* (\varepsilon)[I^{(\beta)}(\Lambda^\varepsilon_{|\beta|}, B)\, I^{(\beta')}(\Gamma^\varepsilon_{|\beta|}, B)]^* (T) \geqslant \varepsilon^{n+\eta}) \text{ is flat.}$$

One dominates the above term by $\sum_{i=1}^3 f^{(i)}_\varepsilon$ where

$$f^{(1)}_\varepsilon = P([B^{(\alpha)} B^{(\alpha')}]^* (\varepsilon) \geqslant \varepsilon^{n-\eta})$$
$$f^{(2)}_\varepsilon = P([I^{(\beta)}(\Lambda^\varepsilon_{|\beta|}, B)]^* (T) \geqslant \varepsilon^{-\eta})$$
$$f^{(3)}_\varepsilon = P([I^{(\beta')}(\Gamma^\varepsilon_{|\beta|}, B)]^* (T) \geqslant \varepsilon^{3\eta}).$$

Since $B^{(\alpha)}, B^{(\alpha')} \in \mathscr{L}_{n/2}$, $\varepsilon \to f_\varepsilon^{(1)}$ is flat. Clearly $f_\varepsilon^{(2)} \leqslant P(\sup_{t \leqslant T} |B^{(\beta)}(t)| \geqslant \varepsilon^{-\eta})$ which is also flat. Finally, to evaluate $f_\varepsilon^{(3)}$, one notes that if $(u_1, ..., u_m) \in \Gamma^\varepsilon_{|\beta'|}$, then at least one of the integration variables $u_1, ..., u_m$ runs in $[0, \varepsilon]$ only. So, by using the Chebyshev inequality first and then the Burkholder inequality one gets for every $p \in N$

$$f_\varepsilon^{(3)} \leqslant \varepsilon^{-3p\eta} E(\sup_{t \leqslant T} |I^{(\beta')}(\Gamma^\varepsilon_{|\beta'|}, B)(t)|^p) \leqslant K\varepsilon^{p(1/2-3\eta)}.$$

Assume that $0 < \eta < 1/6$ (there is no loss of generality in doing it). Then, the above inequality ensures that $\varepsilon \to f_\varepsilon^{(3)}$ is flat and so the whole proof is completed. Q.E.D.

Proof of Lemma 2.6. Let us denote

$$B_\varepsilon(s) = \varepsilon^{-1/2} B(s\varepsilon), \qquad 0 \leqslant s \leqslant 1,$$
$$\bar{B}_\varepsilon(t) = B(t) - B(\varepsilon), \qquad \varepsilon \leqslant t.$$

Notice that

$$I^{(\beta)}(\Lambda^2_{|\beta|}, B)(t) = \bar{B}^{(\beta)}(t - \varepsilon), \qquad t \geqslant \varepsilon,$$
$$B^{(\alpha)}(s) = \varepsilon^{n/2} B_\varepsilon^{(\alpha)}(s/\varepsilon), \qquad 0 \leqslant s \leqslant \varepsilon.$$

It follows that (2.7) reduces to the flatness of

$$\varepsilon \to P\left(\sup_{\varepsilon \leqslant t \leqslant T} \int_0^\varepsilon \left[\sum_{p(\alpha)=n} \sum_{p(\beta)=m} \varepsilon^{n/2} c_{\alpha,\beta} B_\varepsilon^{(\alpha)}(s/\varepsilon)\, \bar{B}_\varepsilon^{(\beta)}(t-\varepsilon)\right]^2 ds \leqslant \varepsilon^{n+1+\eta}\right).$$

Let $u = s/\varepsilon$, $v = t - \varepsilon$. The above term is equal to

$$f_\varepsilon^{(1)} = P\left(\sup_{v \leqslant T-\varepsilon} \int_0^1 \varepsilon^{n+1} \left[\sum_{p(\alpha)=n} \sum_{p(\beta)=m} c_{\alpha,\beta} B_\varepsilon^{(\alpha)}(u)\, \bar{B}_\varepsilon^{(\beta)}(v)\right]^2 du \leqslant \varepsilon^{n+1+\eta}\right).$$

Since the above expression depends on the distribution of $(B_\varepsilon, \bar{B}_\varepsilon)$ only, we may replace B_ε and $\bar{B}_\varepsilon$ by two anonymous independent d-dimensional Brownian motions which we designate by B and $\hat{B}$. Then

$$f_\varepsilon^{(1)} = P\left(\sup_{v \leqslant T-\varepsilon} \int_0^1 Z^2_{n,m}(u, v)\, du \leqslant \varepsilon^\eta\right),$$

where

$$Z(u, v) = \sum_{p(\alpha)=n} \sum_{p(\beta)=m} c_{\alpha,\beta} B^{(\alpha)}(u)\, \hat{B}^{(\beta)}(v).$$

In view of Proposition 2.2 one may find some stopping times $0 \leqslant T_\varepsilon^\eta \leqslant T/2$, $\varepsilon > 0$, such that

$$\varepsilon \to P(\sup_{u \leqslant 1} |Z(u, T_\varepsilon^\eta)| \leqslant 2\varepsilon^{\eta/5}) \text{ is flat.} \tag{2.8}$$

Then one writes

$$f_\varepsilon^{(1)} \leqslant f_\varepsilon^{(2)} = E\left(\int_0^1 Z^2(u, T_\varepsilon^\eta)\, du \leqslant \varepsilon^\eta\right).$$

In order to evaluate $f_\varepsilon^{(2)}$ one uses the same procedure as in the proof of Proposition 2.3. One defines

$$\begin{aligned} t_{\varepsilon,\eta} &= \inf\{0 \leqslant u \leqslant 1/2 \colon |Z(u, T_\varepsilon^\eta)| \geqslant 2\varepsilon^{\eta/5}\} \\ &= \infty \qquad \text{if} \quad \{\cdots\} = \varnothing, \\ t'_{\varepsilon,\eta} &= \inf\{t_{\varepsilon,\eta} \leqslant u \leqslant 1 \colon |Z(u, T_\varepsilon^\eta)| \leqslant \varepsilon^{\eta/5}\} \\ &= \infty \qquad \text{if} \quad \{\cdots\} = \varnothing, \end{aligned}$$

and writes

$$\begin{aligned} f_\varepsilon^{(2)} &\leqslant P(t_{\varepsilon,\eta} = \infty) + P(t_{\varepsilon,\eta} \leqslant \tfrac{1}{2}, t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} \leqslant \varepsilon^{3\eta/5}) \\ &\quad + P\left(t_{\varepsilon,\eta} \leqslant \tfrac{1}{2}, t'_{\varepsilon,\eta} - t_{\varepsilon,\eta} > \varepsilon^{3\eta/5}, \int_0^1 Z^2(u, T_\varepsilon^\eta)\, du \leqslant \varepsilon^\eta\right) \\ &=: f_\varepsilon^{(3)} + f_\varepsilon^{(4)} + f_\varepsilon^{(5)}. \end{aligned}$$

One has $f_\varepsilon^{(5)} = 0$ and (2.8) ensures that $f_\varepsilon^{(3)}$ is flat. Since the family $Z(\cdot, T_\varepsilon^\eta)$, $\varepsilon, \eta > 0$, fulfils $(\hat{L}_{1/2})$, $f_\varepsilon^{(4)}$ is also flat. So the proof of Lemma 2.6 and consequently the proof of Proposition 2.4 is completed. Q.E.D.

3. A Class of Semimartingales

In this section we shall study the following classes of processes:

$$\mathscr{C}_0 = \mathscr{C}$$

$$\mathscr{C}_{k+1} = \left\{ X \colon \text{there are some } X_i \in \mathscr{C}_k,\ 0 \leqslant i \leqslant d, \text{ such that} \right. \\ \left. X(t) - X(0) = \sum_{i=0}^{d} \int_0^t X_i(s)\, dB^i(s) \right\}, \tag{3.1}$$

$$\mathscr{C}_\infty = \bigcap_{k=0} \mathscr{C}_k.$$

One defines $Pr_i: \mathscr{C}_\infty \to \mathscr{C}_\infty$, $0 \leqslant i \leqslant d$, to be

$$Pr_i(X) = X_i, \qquad \text{where} \quad X_i \text{ is that in (3.1).}$$

As X_i, $0 \leqslant i \leqslant d$, in (3.1) are unique up to indistinguishability (and we identify indistinguishable processes) the definition of Pr_i, $0 \leqslant i \leqslant d$, is correct. Next, for a multi-index α one defines

$$Pr_\alpha(X) = Pr_\alpha(Pr_\alpha(X))$$

and

$$pr_\alpha(X) = Pr_\alpha(X)(0).$$

Note that, since for $X \in \mathscr{C}$, $X(0)$ is almost surely constant, $pr_\alpha(X)$ is a real number.

To get unitary notation we put $Pr_\varnothing(X) = X$ and $pr_\varnothing(X) = X(0)$.

Finally, formula (3.1) extends inductively to the Taylor series type formula

$$X = \sum_{p(\alpha) \leqslant k} pr_\alpha(X)\, B^{(\alpha)} + R_k(X), \tag{3.2}$$

where

$$R_k(X) = R'_k(X) + R''_k(X)$$

$$R'_k(X) = \sum_{p(\alpha) = k} I^{(\alpha)}((Pr_\alpha(X) - pr_\alpha(X)), B)$$

$$R''_k(X) = \sum_{p(\alpha) = k-1} I^{(\alpha, 0)}(Pr_\alpha(X), B), \qquad k \geqslant 1.$$

Remark 3.1. $R_k(X) \in \mathscr{L}_{(k+1)/2}$.

Indeed, by (3.1), $Pr_\alpha(X) - pr_\alpha(X) \in \mathscr{L}_{1/2}$ and further, by (2.2)(i), $R'_k(X) \in \mathscr{L}_{(k+1)/2}$. On the other hand, since $p((\alpha, 0)) = p(\alpha) + 2 = k + 1$, $R''_k(X) \in \mathscr{L}_{(k+1)/2}$.

Definition 3.2. For $X \in \mathscr{C}_\infty$ one defines

$$o(X) = \min\{p(\alpha) : pr_\alpha(X) \neq 0\}.$$

The main result in this section is

Theorem 3.3. *Let $X \in \mathscr{C}_\infty$. Then, the following assertions are equivalent*

(i) $o(X) = k$.

(ii) $X \in \mathscr{S}_{k/2} \cap \mathscr{L}_{k/2}$.

(iii) *For every* $0 < u < 1$ *and every* $p \in N$

(a) $\overline{\lim}_{T \to 0} T^{(k+u)/2} E(1/X^*(T)^p)^{1/p} = 0$,

(b) $\underline{\lim}_{T \to 0} T^{(k-u)/2} E(1/X^*(T)^p)^{1/p} = 0$.

(iv) *There exist some* $0 < u < 1$ *and* $p \in N$ *such that* (a) *and* (b) *above hold.*

Remark 3.4. Let us notice that if $X \in \bigcup_{k=0} \mathscr{S}_{k/2}$ then

$$E(1/X^*(T)^p) < \infty \qquad \text{for every} \quad p, T > 0. \tag{3.3}$$

This is because

$$P(X^*(T) \leqslant \varepsilon) \leqslant P(X^*(\varepsilon^{(2/k)-u}) \leqslant \varepsilon) \tag{3.4}$$

which is a flat function if $X \in \mathscr{S}_{k/2}$ (u is a sufficiently small positive number). So $\varepsilon \to P(X^*(T) \leqslant \varepsilon)$ is flat and consequently, in view of Remark 1.2, $1/X^*(T)$ has finite moments of any order.

Inequality (3.4) contains a manifest loss of information: one takes into account the growth of X around $t = 0$ instead of its growth on the whole interval $[0, T]$. This explains why (3.3) cannot be equivalent to $X \in \bigcup_{k=0}^{\infty} \mathscr{S}_{k/2}$ (i.e., $o(X) < \infty$). The gap between these two assertions corresponds to the well-known gap between Hörmander's hypothesis and hypoellipticity.

One the other hand inequality (3.4) suggests that in order to obtain conditions which are closer to (3.3) then $S_{k/2}$) is, one would have to study the same property but around any stopping time $0 \leqslant \tau < T$ instead of $\tau = 0$. (At the level of Hörmander's condition this would correspond to "dim $\mathscr{D}_{\infty, X(\tau)} = n$ a.s. for some stopping time $0 \leqslant \tau < T$," where $\mathscr{D}_{\infty, x}$ is the Lie algebra attached to the differential operator at x.) The difficulty of such an attempt is that $X(\tau)$ would no longer be constant—as $X(0)$ is—and this is a basic assumption in our approach. The same idea is implicit in the Blumenthal zero-one law used by Bismut to solve the covariance matrix problem (see Bismut [1] or Zakai [10]).

Anyway, it is well known that Hörmander's hypothesis is equivalent to hypoellipticity under the supplementary assumption that the coefficients of the differential operator (i.e., of the stochastic equations, in the probabilistic frame) are analytic functions. In the frame here we have the following analogous result.

Let $X \in \mathscr{C}_{\infty}$ be called analytic if $R_k(X) \to 0$ in probability, as $k \to \infty$. For such a process formula (3.2) may be extended to $X = \sum_{p(\alpha)=0}^{\infty} pr_{\alpha}(X) B^{(\alpha)}$ a.s. For an analytic process (3.3) is equivalent to $o(X) < \infty$. This is because if $o(X) = \infty$ then $X = 0$.

Proof of Theorem 3.3. Assume that $o(X)=k<\infty$. Then (3.2) yields

$$X=\sum_{p(\alpha)=k} pr_\alpha(X)\, B^{(\alpha)}+R_k(X). \tag{3.5}$$

By Proposition 2.1, $\sum_{p(\alpha)=k} pr_\alpha(X)\, B^{(\alpha)} \in \mathscr{S}_{k/2}$ and, in view of Remark 3.2, $R_k(X)\in \mathscr{L}_{(k+1)/2}$. Then Proposition 1.4(C) ensures that $X\in \mathscr{S}_{k/2}$. Since all the terms in the right-hand side of (3.5) are in $\mathscr{L}_{k/2}$, so is X. So we have proved (i) $\Rightarrow$ (ii).

Let $X\in \mathscr{S}_{k/2}\cap \mathscr{L}_{k/2}$. If $o(X)\geqslant k+1$ then $X=R_k(X)\in \mathscr{L}_{(k+1)/2}\subseteq \mathscr{S}^c_{k/2}$ and if $o(X)\leqslant k-1$, then the same reasoning as above ensures that $X\in \mathscr{S}_{(k\ \ 1)/2}\subseteq \mathscr{L}^c_{k/2}$. So (ii) $\Rightarrow$ (i) is also proved.

The way to (iii) and (iv) goes through the evaluations contained in the following two lemmas. Let us introduce some notation. For $X\in \mathscr{C}_\infty$ and $p, k\in N$ one denotes

$$H_{p,k}(X)=\max_{r\leqslant k} E\left(\sum_r^* (X)(1)^{-4p(k+1)}\right),$$

where $\sum_k (X)=\sum_{p(\alpha)=k} pr_\alpha(X)\, B^{(\alpha)}$, and

$$Q_{p,k}(X)=\max_{p(\alpha)\leqslant k+3} E([Pr_\alpha^*(X)(1)]^{2p\mu_k}),$$

where $\mu_k=2^{k+2}k(k+1)$.

LEMMA 3.5. *For every $p,k\in N$ there are some constants $K_{p,k}$, C_k, and $C'_k\in(0,\infty)$ such that for every $X\in \mathscr{C}_\infty$ with $o(X)=k$ and every $0< \ <T<1$*

$$E(X^*(T)^{-p})\leqslant \frac{K_{p,k}}{T^{pk/2}}\,[1+H_{p,k}(X)+T^{2p(k+1)}Q_{p,k}(X)+\exp(-C_k/T^{C'_k})]. \tag{3.6}$$

Proof. The proof relies on the following assertion which is (A.5) in Kusuoka and Stroock [6]:

For every $k\in N$ and $u>0$ there are some constants $0<\lambda(k,u)$, $C(k,u)<\infty$ such that for every multi-index α with $p(\alpha)=k$, every $K>0$, and every $Z\in\mathscr{C}_\infty$, one has

$$\begin{aligned} P(&\sup_{0<t\leqslant 1} |I^{(\alpha)}(Z,B)(t)|/t^{k/2\ \ u}\geqslant K^{2^k}, Z^*(1)\leqslant K, Y^*(1)\leqslant K)\\ &\leqslant C(k,u)\exp(-\lambda(k,u)\, K), \end{aligned} \tag{3.7}$$

where $Y(t)=(\sum_{i=1}^d \int_0^t Pr_i(Z)^2\,(s)\,ds)^{1/2}$.

Inequality (3.7) being known, the proof of (3.6) goes as follows. We shall first check that for every $K \geqslant 1$, $0 < T < 1$, and $p \in N$

$$P(X^*(T) \leqslant T^{k/2}/K) \leqslant (K_{p,k}/K^{2p}) \cdot [H_{p,k}(X) + T^{2pk(k+1)}Q_{p,k}(X)] \\ + \exp(-C_k/T^{C'_k}) \exp(-C''_k K^{1/u_k}), \tag{3.8}$$

where $K_{p,k}$ is a constant depending on $k, p \in N$ and C_k, C'_k, C''_k are some constants depending on $k \in N$.

Let $\gamma = (2k+1)/k(k+1)$. In view of (3.3) one has

$$\begin{aligned} P(X^*(T) \leqslant T^{k/2}/K) &\leqslant P(X^*(T/K^\gamma) \leqslant T^{k/2}/K) \\ &\leqslant P\Big(\sum_k^* (X)(T/K^\gamma) \leqslant 2T^{k/2}/K\Big) \\ &\quad + P(R_k^*(X)(T/K^\gamma) \geqslant T^{k/2}/K) =: I + J. \end{aligned}$$

By using the scale property (2.1) one gets

$$\begin{aligned} I &= P\Big((T/K^\gamma)^{k/2} \sum_k^* (X)(1) \leqslant 2T^{k/2}/K\Big) \\ &= P\Big(\sum_k^* (X)(1) \leqslant 2K^{-1/2(k+1)}\Big) \\ &= P\Big(\sum_k^* (X)(1)^{-4p(k+1)} \geqslant K^{2p}/2^{4p(k+1)}\Big) \\ &\leqslant (2^{4p(k+1)}/K^{2p})\, H_{p,k}(X). \end{aligned}$$

Let us then notice that

$$\begin{aligned} R_k(X) = \sum_{i=0}^{d} \sum_{p(\alpha)=k} & I^{(\alpha,i)}(Pr_{(\alpha,i)}(X), B) \\ + \sum_{p(\alpha)=k-1} & I^{(\alpha,0)}(Pr_\alpha(X), B), \end{aligned}$$

so, $R_k(X)$ is a finite sum of terms of the form $I^{((\beta)}(Pr_\alpha(X), B)$ with $p(\beta) = k+1, k+2$ and $p(\alpha) = k-1, k+1, k+2$. Consequently it will suffice to evaluate the probabilities of type J for terms of this form instead of $R_k(X)$. Let us fix α and β and denote $U = I^{(\beta)}(Pr_\alpha(X), B)$, $Z = Pr_\alpha(X)$, $Y = (\sum_{i=1}^d \int_0 Pr_i(Z)^2\,(s)\,ds)^{1/2}$. One writes (for β with $p(\beta) = k+1$; for β with $p(\beta) = k+2$ the reasoning is the same):

$$P(U^*(T/K^\gamma) \geqslant T^{k/2}/K) = P\Big(U^*(T/K^\gamma)/(T/K^\gamma)^{(k+1)/2-1/4} \\ \geqslant \frac{T^{k/2}}{K} \cdot \Big(\frac{K^\gamma}{T}\Big)^{(k+1)/2-1/4}\Big).$$

Since $\gamma((k+1)/2-1/4)-1=1/4k(k+1)$, the above term is equal to

$$\begin{aligned}&P(U^*(T/K^\gamma)/(T/K^\gamma)^{(k+1)/2-1/4}\geqslant K^{1/4k(k+1)}/T^{1/4})\\&\quad\leqslant P(\sup_{t\leqslant 1}|U(t)|/t^{(k+2)/2-1/4}\geqslant K^{1/4k(k+1)}/T^{1/4}).\end{aligned}$$

Let us denote $\bar{K}=(K^{1/4k(k+1)}/T^{1/4})^{1/2^k}=K^{1/u_k}\,T^{1/2^{k+2}}$ and dominate the above term by

$$\begin{aligned}&P(\sup_{t\leqslant 1}|U(t)|/t^{(k+1)/2-1/4}\geqslant\bar{K}^{2^k},\,Z^*(1)\leqslant\bar{K},\,Y^*(1)\leqslant\bar{K})\\&\quad+P(Z^*(1)\geqslant\bar{K})+P(Y^*(1)\geqslant\bar{K})=:J'+J''+J'''.\end{aligned}$$

By (3.7)

$$\begin{aligned}J'&\leqslant C(k,\tfrac{1}{4})\exp(-\lambda(k,\tfrac{1}{4})\,k^{1/\mu_k}/T^{1/2^{k+2}})\\&\leqslant C(k,\tfrac{1}{4})\exp(-C_k'/T^{C_k})\exp(-C_k'K^{1/u_k}),\end{aligned}$$

where $C_k'=\lambda(k,\frac{1}{4})/2$ and $C_k=1/2^{k+2}$.

Let us now evaluate J'' (for J''' the computations are alike). For every $p\in N$ one has

$$\begin{aligned}J''&=P(Z^*(1)^{2p\mu_k}\geqslant\bar{K}^{2p\mu_k})\leqslant\bar{K}^{2p\mu_k}E([Pr_x^*(X)(1)]^{2p\mu_k})\\&\leqslant(T^{2pk(k+1)}/K^{2p})\,Q_{p,k}(X).\end{aligned}$$

So (3.8) is proved and we are ready to prove (3.6):

$$\begin{aligned}E(1/X^*(T)^p)&\leqslant\sum_{h=0}^{\infty}\left(\frac{h+1}{T^{k/2}}\right)^p P\left(\frac{h}{T^{k/2}}\leqslant\frac{1}{X^*(T)}<\frac{h+1}{T^{k/2}}\right)\\&\leqslant T^{-kp/2}+\sum_{h=0}^{\infty}(h+1)^p\,T^{-kp/2}P(X^*(T)\leqslant h/T^{k/2})\\&\leqslant T^{-kp/2}\Bigg[1+(H_{p,k}(X)+T^{2pk(h+1)}Q_{p,k}(X))\\&\quad\cdot\left(\sum_{h=1}^{\infty}(h+1)^p/h^{2p}\right)\\&\quad+\exp(-C_k/T^{C_k})\left(\sum_{h=1}^{\infty}(h+1)^p\exp(-C_k''h^{1/\mu_k})\right)\Bigg].\end{aligned}$$

Since $\sum_{h=1}^{\infty}(h+1)^p/h^{2p}<\infty$ and $\sum_{h=1}^{\infty}(h+1)^p\exp(-C_k''h^{1/\mu_k})<\infty$ depend on p and k only, the proof of (3.4) is completed. Q.E.D.

We go on and establish a minoration:

LEMMA 3.6. *Let $X \in \mathscr{C}_\infty$ with $o(X) \geqslant k$. Then, for every $p \in N$, $0 < T \leqslant 1$, and $u > 0$*

$$E(X^*(T)^{-p})^{1/p} \geqslant [K(k, u) \wedge (8e_k Q_{1,k-1}(X))^{-2^k}]/2T^{(k-u)/2}, \tag{3.9}$$

where the constants in the above formula are defined as

$$e_k = (h_k + h_{k+1})(h_{k-2} + h_k + h_{k+1}) \qquad \textit{with} \quad h_k = \# \{\alpha: p(\alpha) = k\},$$

$$K(k, u) = [\lambda(k, u)/\ln(4e_k C(k, u))]^{2^k},$$

$\lambda(k, u)$ and $C(k, u)$ being the constants in (3.7).

Proof. Let us take some $K > 0$ (to be made precise later) and write

$$\begin{aligned} E(X^*(T)^{-p}) &\geqslant E(X^*(T)^{-p}, X^*(T) \leqslant KT^{(k-u)/2}) \\ &\geqslant K^{-p} T^{-p(k-u)/2} P(X^*(T) \leqslant KT^{(k-u)/2}) \\ &= K^{-p} T^{-p(k-u)/2} (1 - P(X^*(T) \geqslant KT^{(k-u)/2})). \end{aligned} \tag{3.10}$$

We have to choose K such that $P(X^*(T) \geqslant KT^{(k-u)/2}) \leqslant \frac{1}{2}$. Since $o(X) \geqslant k$ one has $X = R_{k-1}(X)$ and so, what is to be proved is

$$P(U^*(T) \geqslant KT^{(k-u)/2}) \leqslant 1/2e_k, \tag{3.11}$$

where $U = I^{(\beta)}(Pr_\alpha(X), B)$, $p(\beta) = k, k+1$, and $p(\alpha) = k-2, k+1$. As in the proof of Lemma 3.5 we denote $Z = Pr_\alpha(X)$ and $Y = (\sum_{i=1}^d \int_0 Pr_i(Z)^2 (s)\, ds)^{1/2}$ and use (3.7) to get

$$\begin{aligned} P(U^*(T) \geqslant KT^{(k-u)/2}) &= P(U^*(T)/T^{(k-u)/2} \geqslant K) \\ &\leqslant P(\sup_{t \leqslant 1} |U(t)|/t^{(k-u)/2} \geqslant K, Z^*(1) \\ &\leqslant K^{1/2^k}, Y^*(1) \leqslant K^{1/2^k}) \\ &\quad + P(Z^*(1) \geqslant K^{1/2^k}) + P(Y^*(1) \geqslant K^{1/2^k}) \\ &\leqslant C(k, u) \exp(-\lambda(k, u)\, K^{1/2^k}) + 2K^{-1/2^k} Q_{1,k-1}(X). \end{aligned}$$

If $K \geqslant 1/K(k, u)$, the first term is dominated by $1/4e_k$ and if $K \geqslant [8e_k Q_{1,k-1}(X)]^{2^k}$ the second term is dominated by $1/4e_k$ and consequently (3.11) holds. So one takes K to be the maximum of the two above values. Now (3.10) with the above K yields (3.9). Q.E.D.

Now the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) in Theorem 3.3 are obvious and so the whole proof of the theorem is completed. Q.E.D.

4. The Covariance Matrix

In this section we shall deal with a matrix with elements in $\mathscr{C}_\infty$. As will be clear in the following paragraph this matrix has the same form as the one which appears when applying Malliavin calculus to a diffusion. Unfortunately the result we prove here does not work for the solution of stochastic equations with coefficients depending on the past. The reason is that the matrix we work with has the form one obtains for the Malliavin covariance matrix after applying the variance of constants method which does not work in the non-Markov case.

For a space E ($\mathscr{C}_\infty$ or R in our case) we denote by $\mathscr{M}_{n\times m}(E)$ the matrices with n rows and m columns and by E^n the n-dimensional column vectors. For $A \in \mathscr{M}_{n\times m}(E)$ we denote by $\hat{A}$ the transposed matrix, i.e., $\hat{A}^i_j = A^j_i$ and A^i (resp. A_j) will be the i-row (resp. the j-column) of the matrix. The operators Pr_α and pr_α work on components.

For a matrix $U \in \mathscr{M}_{n\times m}(\mathscr{C}_\infty)$ one defines

$$S_k(U) = \operatorname{Span}\{pr_\alpha(U_j) : 1 \leqslant j \leqslant m,\ 0 \leqslant p(\alpha) \leqslant k\} \subseteq R^n, \qquad k \in N,$$

$$S_\infty(U) = \bigcup_{k=0}^{\infty} S_k(U),$$

$$\begin{aligned} h(U) &= \min\{k : \dim S_k(U) = n\} \\ &= \infty \qquad \text{if} \quad \{\cdots\} = \varnothing, \text{ i.e., } \dim S_\infty(U) < n. \end{aligned}$$

We shall denote by $\int U\hat{U}$ the matrix defined by

$$\left(\int U\hat{U}\right)^i_j (t) = \int_0^t (U\hat{U})^i_j (s)\, ds$$

and put

$$\Delta_U(t) = \det\left(\int U\hat{U}\right)(t).$$

The main result in this section is

Theorem 4.1. *Let $U \in \mathscr{M}_{n\times m}(\mathscr{C}_\infty)$. Then:*

(i) $h(U) \leqslant o(\Delta_U) \leqslant 2n(h(U)+1)$.

(ii) *For every $p \in N$, $0 < T \leqslant 1$, $0 < u$*

$$\lambda/T^{(h(U)-u)/2} \leqslant E(1/\Delta_U(T)^p)^{1/p} \leqslant \lambda'/T^{n(h(U)+1)},$$

where

$$\lambda = \tfrac{1}{2}\left[K(h(U), u) \wedge (8e_{2n(h(U)+1)}Q_{1,2n(h(U)+1)}(\Delta_U))^{-2^{h(U)}}\right]$$

and

$$\lambda' = K_{p,U}[1 + H_{p,2n(h(U)+1)}(\Delta_U) + T^{2pn(h(U)+1)}Q_{p,2n(h(U)+1)}(\Delta_U) + \exp(-C_U/T^{C'_U})]^{1/p},$$

where $Q_{1,k}(\Delta_U)$, $H_{p,k}(\Delta_U)$ *are defined at the beginning of the previous section,* $K(h(U), u)$ *and* e_k *are those in Lemma* 3.6, *and* $K_{p,U}$, C_U, *and* C'_U *are constants depending on* p *and* $h(U)$.

(iii) *The following assertions are equivalent:*

(a) $\dim S_\infty(U) = n$.

(b) $\Delta_U \in \bigcup_{k=0}^{\infty} \mathscr{S}_k$.

(c) $o(\Delta_U) < \infty$.

(d) *There is some* $k \in N$ *such that for every* $p \in N$

$$\lim_{T \to 0} T^k E(1/\Delta_U(T)^p)^{1/p} = 0.$$

(e) *There are some* $k \in N$ *and* $p \in N$ *such that*

$$\lim_{T \to 0} T^k E(1/\Delta_U(T)^p)^{1/p} = 0.$$

Proof. We claim that (ii) and (iii) are easy consequences of (i) and of the results in Section 3. To check this one notices that $t \to \Delta_U(t)$ is non-decreasing and so $\Delta_U(T) = \Delta_U^*(T)$. Then the inequalities in (ii) are immediate consequences of (i) and of Lemma 3.5 and Lemma 3.6.

Let us have a look at (iii). The equivalence between (b), (c), (d), (e) is an immediate consequence of Theorem 3.3 and (a) $\Leftrightarrow$ (c) is a consequence of (i).

The proof of (i) proceeds in steps. The main one is

LEMMA 4.2. *Let* $q \in N$. *The following assertions are equivalent:*

(i) $\dim S_q(U) = n$.

(ii) $o(\langle (\int U\hat{U}), x, x \rangle) \leqslant 2(q+1) + o(\langle x, x \rangle)$ *for every* $x \in \mathscr{C}^n_\infty$,

where $\langle\ ,\ \rangle$ *designates the usual scalar product in* R^n.

Proof. We shall first prove (i) $\Rightarrow$ (ii). Let $x \in \mathscr{C}^n_\infty$ and

$$2p = o(\langle x, x \rangle) = o\left(\sum_{i=1}^{n} (x^i)^2\right) = 2 \min_{1 \leqslant i \leqslant n} o(x^i).$$

The last equality is obtained in the following way: since $(x^i)^2$, $1 \leqslant i \leqslant n$, are positive processes, if at least one of them fulfils a property of type (S_h), then the sum fulfils the same property.

Let β be such that $p(\beta)=p$ and $pr_\beta(x)\neq 0$ (i.e., at least one of $pr_\beta(x^i)\neq 0$, $1\leqslant i\leqslant n$). Since $\dim S_q(U)=n$ there is some α with $p(\alpha)\leqslant q$ and $1\leqslant j\leqslant m$ such that $\langle pr_\alpha(U_j), pr_\beta(x)\rangle\neq 0$ and consequently

$$[pr_\beta(\hat{U})\, pr_\alpha(x)]^j=\langle pr_\alpha(U^j), pr_\beta(x)\rangle\neq 0.$$

One concludes that

$$k_0=:\min\{p(\alpha)+p(\beta): pr_\alpha(\hat{U})\, pr_\beta(x)\neq 0\}\leqslant q+p=q+\tfrac{1}{2}o(\langle x, x\rangle).$$

In view of Theorem 3.3, in order to prove (ii) we have to check that $\langle(\int U\hat{U})\, x, x\rangle\in\mathscr{S}_{k_0+1}$. Notice that

$$\left\langle\left(\int U\hat{U}\right)_t x_t, x_t\right\rangle=\int_0^t \langle(U\hat{U})_s\, x_t, x_t\rangle\, ds=\int_0^t \|\hat{U}_s\, x_t\|^2\, ds.$$

We take $N=2k_0+1$ and develope U and x in Taylor series of order N (see (3.2)). To simplify the notation we put $U_\alpha=pr_\alpha(U)$ and $x_\beta=pr_\beta(x)$:

$$\hat{U}(s)=\sum_{p(\alpha)\leqslant N}\hat{U}_\alpha B^{(\alpha)}(s)+R_N(\hat{U})(s),$$

$$x(t)=\sum_{p(\beta)\leqslant N} x_\beta B^{(\beta)}(t)+R_N(x)(t).$$

By making their product one gets

$$\begin{aligned}\hat{U}(s)\, x(t)=&\left[\sum_{k_0\leqslant p(\alpha)+p(\beta)\leqslant 2N}\hat{U}_\alpha x_\beta B^{(\alpha)}(s)\, B^{(\beta)}(t)\right]\\ &+\Bigg[R_N(\hat{U})(s)\, R_N(x)(t)+\left(\sum_{p(\alpha)\leqslant N}\hat{U}_\alpha B^{(\alpha)}(s)\right)R_N(x)(t)\\ &+R_N(\hat{U})(s)\sum_{p(\beta)\leqslant N} x_\beta B^{(\beta)}(t)\Bigg]=: Q(s,t)+Q'(s,t).\end{aligned}$$

Let us fix $0<\eta<1/4$. Since $\|Q+Q'\|^2\geqslant\|Q\|^2-2\,\|Q\|\cdot\|Q'\|$, one has

$$\begin{aligned}&P\left(\sup_{t\leqslant\varepsilon}\int_0^t \|U_s x_t\|^2\, ds\leqslant\varepsilon^{k_0+1+\eta}\right)\\ &\quad\leqslant P\left(\sup_{t\leqslant\varepsilon}\int_0^t \|Q(s,t)\|^2\, ds\leqslant 2\varepsilon^{k_0+1+\eta}\right)\\ &\qquad+P\left(\sup_{t\leqslant\varepsilon}\int_0^t 2\,\|Q(s,t)\|\cdot\|Q'(s,t)\|\, ds\geqslant\varepsilon^{k_0+1+\eta}\right)=: f_\varepsilon^{(1)}+f_\varepsilon^{(2)}.\end{aligned}$$

One has

$$
\begin{aligned}
f_\varepsilon^{(2)} &\leqslant P\big(\sup_{s,t\leqslant\varepsilon} \|Q(s,t)\|\cdot\|Q'(s,t)\| \geqslant \tfrac{1}{2}\varepsilon^{k_0+\eta}\big)\\
&\leqslant P\big(\sup_{s,t\leqslant\varepsilon} \|Q(s,t)\| \geqslant \varepsilon^{-\eta}\big)\\
&\quad + P\big(\sup_{s,t\leqslant\varepsilon} \|Q'(s,t)\| \geqslant \tfrac{1}{2}\varepsilon^{k_0+2\eta}\big) =: f_\varepsilon^{(3)} + f_\varepsilon^{(4)}.
\end{aligned}
$$

Since $E(\sup_{s,t\leqslant\varepsilon}\|Q(s,t)\|^p)<\infty$ for every $p\in N$, the function $\varepsilon\to f_\varepsilon^{(3)}$ is flat. On the other hand, since all the elements in $R_N(U)$ and $R_N(x)$ fulfil $(L_{k_0+1/2})$ (see Remark 3.1), $\varepsilon\to f_\varepsilon^{(4)}$ is also flat.

Let us now evaluate $f_\varepsilon^{(1)}$. Denote

$$
Q_k(s,t)=\sum_{p(\alpha)+p(\beta)=k} \hat{U}_\alpha x_\beta B^{(\alpha)}(s)\, B^{(\beta)}(t).
$$

Then,

$$
Q(s,t)=\sum_{k=k_0}^{2N} Q_k(s,t).
$$

By using the scale property (2.1), $Q_k(s,t)\sim\varepsilon^{k/2}Q_k(s/\varepsilon,t/\varepsilon)$ (by "$\sim$" we mean that the two processes—as processes of s and t—are identically distributed) and consequently

$$
\begin{aligned}
&\sup_{t\leqslant\varepsilon}\int_0^t \Big\|\sum_{k=k_0}^{2N} Q_k(s,t)\Big\|^2 ds\\
&\quad\sim \sup_{t\leqslant\varepsilon}\int_0^t \Big\|\sum_{k=k_0}^{2N} \varepsilon^{k/2}Q_k(s/\varepsilon,t/\varepsilon)\Big\|^2 ds\\
&\quad= \sup_{t\leqslant 1}\int_0^t \Big\|\sum_{k=k_0}^{2N} \varepsilon^{k/2}Q_k(s,t)\Big\|^2\cdot\varepsilon\, ds\\
&\quad= \varepsilon^{k_0+1}\sup_{t\leqslant 1}\int_0^t \Big\|Q_{k_0}(s,t)+\sum_{k=k_0+1}^{2N}\varepsilon^{(k-k_0)/2}Q_k(s,t)\Big\|^2 ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
f_\varepsilon^{(1)} &= P\Big(\sup_{t\leqslant 1}\int_0^t \Big\|Q_{k_0}(s,t)+\sum_{k=k_0+1}^{2N}\varepsilon^{(k-k_0)/2}Q_k(s,t)\Big\|^2 ds\leqslant 2\varepsilon^\eta\Big)\\
&\leqslant P\Big(\sup_{t\leqslant 1}\int_0^t \|Q_{k_0}(s,t)\|^2\, ds\leqslant 4\varepsilon^\eta\Big)\\
&\quad + P\Big(\sup_{t\leqslant 1}\int_0^t 2\cdot\|Q_{k_0}(s,t)\|\\
&\quad\cdot\Big\|\sum_{k=k_0+1}^{2N}\varepsilon^{(k-k_0)/2}Q_k(s,t)\Big\|\, ds\geqslant 2\varepsilon^\eta\Big) =: f_\varepsilon^{(5)}+f_\varepsilon^{(6)}.
\end{aligned}
$$

Since $(k-k_0)/2 \geqslant \frac{1}{2} > \eta$ the function $\varepsilon \to f_\varepsilon^{(6)}$ is flat.
Write then

$$f_\varepsilon^{(5)} \leqslant \min_{1 \leqslant i \leqslant m} P\left(\sup_{t \leqslant 1} \int_0^t (Q_{k_0}^i(s,t))^2 \, ds \leqslant 4\varepsilon^\eta\right) =: \min_{1 \leqslant i \leqslant m} g_\varepsilon^{(i)}.$$

Since at least one of the sums

$$Q_{k_0}^i(s,t) = \sum_{p(\alpha)+p(\beta)=k_0} (\hat{U}_\alpha x_\beta)^i \, B^{(\alpha)}(s) \, B^{(\beta)}(t), \qquad 1 \leqslant i \leqslant m,$$

has a non-null coefficient, Proposition 2.4 ensures that at least one of $\varepsilon \to g_\varepsilon^{(i)}$, $1 \leqslant i \leqslant m$, is flat. Then so is $\varepsilon \to f_\varepsilon^{(5)}$. So the proof of (i) $\Rightarrow$ (ii) is completed.

To prove (ii) $\Rightarrow$ (i) one assumes that $\dim S_q(U) < n$. Then, for at least one non-null $x \in R^n$ one has $x \perp S_q(U)$ and consequently $\hat{U}x = R_q(U)\, x \in \mathscr{L}_{(q+1)/2}$. Then $t \to \int_0^t \|U_s x\|^2 \, ds$ fulfils (L_{q+2}) and consequently $o(\langle(\int U\hat{U})x, x\rangle) \geqslant 2q+4 = 2q+4+o(\langle x,x\rangle)$ ($x \neq 0$ implies $o(\langle x,x\rangle)=0$) which contradicts (ii). Q.E.D.

The way from Lemma 4.2 to (i) in Theorem 4.1 goes through the following algebra lemma:

LEMMA 4.3. *Let I be a commutative ring and $o\colon I \to N \cup \{\infty\}$ a function which is not identically infinite and such that*

$$o(xy) = o(x) + o(y), \qquad x, y \in I.$$

Let $a \in \mathscr{M}_{n \times m}(I)$. Assume that for some $k \in N$ one has

$$o(\langle ax, x\rangle) \leqslant 2 \min_{1 \leqslant i \leqslant n} o(x^i) + k \qquad \text{for every} \quad x \in I^n. \tag{4.1}$$

Then $o(\det a) \leqslant kn$.

Proof. We proceed by induction over n. For $n=1$, $a, x \in I$. Take some $x \in I$ such that $o(x) < \infty$. Then

$$o(a) + 2o(x) = o(ax^2) = o(\langle ax, x\rangle) \leqslant 2o(x) + k$$

and so $o(a) \leqslant k$.

Assume that the assertion is proved for $n-1$ and let us check it for n. Let $a = (a_j^i)_{1 \leqslant j \leqslant n}^{1 \leqslant i \leqslant n}$. We denote

$$A_j^i = (a_p^k)_{p \neq j}^{k \neq i} \in \mathscr{M}_{n-1,n-1}(I) \qquad \text{and} \qquad \Gamma_j^i = (-1)^{i+j} \det(A_j^i), \; 1 \leqslant i, j \leqslant n.$$

Let us check that A_n^n fulfils (4.1). Notice that $o(0)=\infty$. Take then $x \in I^{n-1}$ and put $\bar{x}=(x^1, ..., x^{n-1}, 0)$. One has $\langle A_n^n x, x\rangle = \langle ax, x\rangle$ and so, by (4.1) for the matrix a

$$o(\langle A_n^n x, x\rangle) \leqslant 2 \min_{1 \leqslant i \leqslant n} o(x^i)+k = 2 \min_{1 \leqslant i \leqslant n} o(x^i)+k.$$

So A_n^n fulfils (4.1) and consequently, by the induction hypothesis, $o(\Gamma_n^n)=o(\det A_n^n) \leqslant k(n-1)$ (notice that $o(1)=o(-1)=0$).

Let us now take $x=(0, ..., 0, 1)$ and apply (4.1) to the vector $\bar{a}x$ where $\bar{a}$ is the transposed of $(\Gamma_j^i)_{1 \leqslant j \leqslant n}^{1 \leqslant i \leqslant n}$. Notice $\bar{a}x=(\Gamma_n^1, ..., \Gamma_n^n)$ and, since $a\bar{a}=Ix \det a$, $a(\bar{a}x)=(a\bar{a})\,x=(\det a)\cdot x$, it follows that

$$\langle a(\bar{a}x), \bar{a}x\rangle = \langle (\det a)\cdot x, \bar{a}x\rangle = (\det a)\,\Gamma_n^n.$$

Then, by (4.1)

$$o(\det a)+o(\Gamma_n^n)=o(\langle a(\bar{a}x), \bar{a}x\rangle) \leqslant 2 \min_{1 \leqslant i \leqslant d} o((\bar{a}x)^i)+k \leqslant 2o(\Gamma_n^n)+k,$$

which, by the induction hypothesis, yields $o(\det a) \leqslant o(\Gamma_n^n)+k \leqslant nk$.

Q.E.D.

We are now able to prove (i) in Theorem 4.1. The second inequality is an immediate consequence of Lemma 4.2 and Lemma 4.3. To prove the first inequality one takes $k<h(U)$ (i.e., $\dim S_k(U)<n$) and chooses a non-null $x \in R^n$ such that $x \perp S_k(U)$. Consequently $o((Ux)^i)=o(\langle U_i, x\rangle) \geqslant k$, $1 \leqslant i \leqslant m$. It follows that $o([(\int U\hat{U})\,x]^i) \geqslant k+1$. Let $a=\int U\hat{U}$ and let $\bar{a}$ be the matrix defined in the proof of Lemma 4.3. Then

$$o(\det a)=o((\det a)\langle x, x\rangle)=o(\langle \bar{a}ax, x\rangle) \geqslant k+1.$$

This yields $o(\det a) \geqslant h(U)$ and so the proof of (i) in Theorem 4.1 and consequently the whole proof is completed. Q.E.D.

5. The Covariance Matrix Associated to a Non-homogeneous Diffusion Process

In this section we consider a non-homogeneous diffusion process and apply the results from Section 4 to evaluate the determinant of the covariance matrix attached to the diffusion in Malliavin calculus. More exactly, we get several characterizations of the above determinant (which is a process in $\mathscr{C}_\infty$ itself) which are equivalent to Hörmander's condition, on the one hand, and are very close to the assumption on the covariance matrix in Malliavin's absolute continuity theorem, on the other hand.

Let $C_b^\infty(R_+ \times R^n, R^n)$ be the space of the infinitely differentiable functions from $R_+ \times R^n$ to R^n which are bounded and have bounded derivatives of any order and let $\varphi_j \in C_b^\infty(R_+ \times R^n, R^n)$, $0 \leqslant j \leqslant d$.

Consider the stochastic equation

$$dX(t) = \sum_{j=1}^{d} \varphi_j(t, X(t))\, dB^j(t) + \varphi_0(t, X(t))\, dt,$$
$$X(0, x, \omega) = x, \tag{5.1}$$

where dB^j, $1 \leqslant j \leqslant d$, designates the Itô integral.

In order to write the above equation in the Fisk–Stratonovich form we introduce the following notations: for $f \in C_b^\infty(R_+ \times R^n, R^n)$ we denote by $\dot{f}_x$ the matrix $(f_x)_j^i = \partial_x^j f^i$, where $\partial_x^j = \partial/\partial x^j$, $1 \leqslant j \leqslant d$. Then one denotes

$$\bar{\varphi}_0 = \varphi_0 - \sum_{j=1}^{d} \dot{\varphi}_{j,x} x \varphi_j$$

and writes Eq. (5.1) in the form

$$dX(t) = \sum_{j=1}^{d} \varphi_j(t, X(t)) \circ dB^j(t) + \bar{\varphi}_0(t, X(t))\, dt,$$
$$X(0, x, \omega) = x, \tag{5.2}$$

where $\circ\, dB^j$ designates the Fisk–Stratonovich integral.

We shall also be interested in the equations

$$dY(t) = \sum_{j=1}^{d} \dot{\varphi}_{j,x}(t, X(t))\, Y(t) \circ dB^j(t) + \dot{\bar{\varphi}}_{0,x}(t, X(t))\, dt,$$
$$Y(0, \omega) = I \text{ (the identity matrix)}, \tag{5.3}$$

and

$$dZ(t) = -\sum_{j=1}^{d} Z(t)\, \dot{\varphi}_{j,x}(t, X(t)) \circ dB^j(t) - Z(t)\, \dot{\bar{\varphi}}_{0,x}(t, X(t))\, dt,$$
$$Z(0, \omega) = I. \tag{5.4}$$

As is well known $Y(t, \omega)$ represents the derivative of $x \to X(t, x, \omega)$ and Z is the inverse of Y, i.e., $Y(t)\, Z(t) = Z(t)\, Y(t) = I$ for every $t \geqslant 0$ a.s. It is also known that each element of Y and Z has moments of any order and, since $1/\det Y(t) = \det Z(t)$, this yields $E(\sup_{t \leqslant T} |1/\det Y(t)|^p) < \infty$ for every p, $T > 0$.

Let us now fix some $x \in R^n$, $t > 0$, and think of $X(t, x, \omega)$ as a functional of the Brownian motion. Then, the covariance matrix attached in Malliavin calculus to $X(t, x, \omega)$ is

$$\sigma_x(t) = \int_0^t [Y_t Z_s \varphi(s, X_s)] \cdot [Y_t Z_s \varphi(s, X_s)]^\wedge \, ds,$$

which is a process in $\mathscr{C}_x$.

The above formula is obtained in two steps: one first checks that the Malliavin derivatives of $X(t, x, \omega)$ fulfil a certain stochastic equation and then one uses the variance of constants method to get the above form of $\sigma_x(t)$ (see, e.g., Ikeda Watanabe [4], Watanabe [9], Kusuoka and Stroock [5], and so on).

The difficulty in applying Malliavin calculus to $X(t, x, \omega)$ is to check that $E(1/\det \sigma_x(t)^p) < \infty$ for every $p \in N$. In view of what was told above about Y this reduces to

$$E\left(1/\det\left(\int U_x \hat{U}_x\right)(t)^p\right) < \infty \qquad \text{for every} \quad p \in N, \tag{5.5}$$

where

$$U_x(t, \omega) = Z(t, \omega)\, \varphi(t, X(t, x, \omega)).$$

We also notice that, since $o(\det Y) = o(\det Y(0) = 1)$, one has

$$o(\det \sigma_x) = o\left(\det\left(\int U_x \hat{U}_x\right)\right). \tag{5.6}$$

We have to describe $S_k(U_x)$, $k \in N$ (see Sect. 4). To do it we introduce the following notation. Let $T_j, T'_j \colon C_b^\infty(R_+ \times R^n, R^n) \to C_b^\infty(R_+ \times R^n, R^n)$, $0 \leqslant j \leqslant d$, be defined by

$$T_j f = T'_j f = [\varphi_j, f]_x, \qquad 1 \leqslant j \leqslant d,$$

$$T_0 f = [\bar{\varphi}_0, f]_x + \partial_t f$$

$$T'_0 f = [\bar{\varphi}_0, f]_x + \partial_t f + \sum_{j=1}^d [\varphi_j, [\varphi_j, f]_x]_x,$$

where $[\ ,\ ]_x$ designates the Lie brackets with respect to the variable x, that is,

$$[f, g]_x^i\,(t, x) = \sum_{k=1}^n (f^k(t, x)\, \partial_x^k g^i(t, x) - g^k(t, x)\, \partial_x^k f^i(t, x)).$$

For a multi-index $\alpha = (\alpha_1, ..., \alpha_r)$, $0 \leqslant \alpha_i \leqslant d$, one defines $T_\alpha = T_{\alpha_r} \circ \cdots \circ T_{\alpha_1}$ and $T' = T'_{\alpha_r} \circ \cdots \circ T'_{\alpha_1}$ and for $\alpha = \varnothing$ one takes $T_\varnothing = T'_\varnothing =$ the identity application.

Define then

$$\mathscr{D}_k = \{ T_\alpha(\varphi_j) : 1 \leqslant j \leqslant d, p(\alpha) \leqslant k \}, \qquad \mathscr{D}_\infty = \bigcup_{k=0}^{\infty} \mathscr{D}_k,$$

$$\mathscr{D}'_k = \{ T'_\alpha(\varphi_j) : 1 \leqslant j \leqslant d, p(\alpha) \leqslant k \}, \qquad \mathscr{D}'_\infty = \bigcup_{k=0}^{\infty} \mathscr{D}'_k.$$

Clearly

$$\mathscr{D}'_k \subseteq \mathscr{D}_{k+1} \subseteq \mathscr{D}'_{k+2}, \qquad k \in N. \tag{5.7}$$

We shall also denote

$$\mathscr{D}_{k,x} = \mathrm{Span}\{ (f(x) : f \in \mathscr{D}_k \} \qquad \text{and} \qquad \mathscr{D}'_{k,x} = \mathrm{Span}\{ (f(x) : f \in \mathscr{D}'_k \}.$$

Then

$$S_k(U_x) = \mathscr{D}'_{k,x}, \tag{5.8}$$

where $S_k(U_x)$ is defined in Section 4. The above equality is an immediate consequence of the following remark: for every $f \in C_b(R_+ \times R^n, R^n)$ one has

$$dZ(t) f(t, X(t)) = \sum_{j=1}^{d} Z(T)\, T_j(f)(t, X(t)) \circ dB^j(t) + Z(t)\, T_0(f)(t, X(t))\, dt$$

which in Itô form is

$$dZ(t) f(t, X(t)) = \sum_{j=1}^{d} Z(t)\, T'_j(f)(t, X(t))\, dB^j(t) + Z(t)\, T'_0 f(t, X(t))\, dt. \tag{5.9}$$

Let us denote

$$h(x) = \min\{ k : \dim \mathscr{D}_{k,x} = n \}$$
$$= \infty, \qquad \text{if} \quad \{ \cdots \} = \varnothing \text{ (i.e., } \dim \mathscr{D}_{\infty,x} < n).$$

Then, in view of (4.7)

$$h(x) \leqslant h(U_x) + 1 \leqslant h(x) + 2. \tag{5.10}$$

The above inequality is interesting because $h(x) < \infty$ represents the Hörmander condition.

Before stating our result we have to introduce

$$K_r(\varphi) = \max_{|\alpha| \leqslant r} \max_{1 \leqslant i \leqslant n} \max_{0 \leqslant j \leqslant d} \sup_{(t,x) \in R_+ \times R^n} |D^{(\alpha)} \varphi_j^i(t,x)|,$$

where $D^{(\alpha)} = \partial^{\alpha_k} \cdots \partial^{\alpha_1}$ for $\alpha = (\alpha_1, ..., \alpha_k)$. We have put $\partial^i = \partial / \partial x^i$ for $1 \leqslant i \leqslant d$ and $\partial^0 = \partial / \partial t$.

THEOREM 5.1. *For each* $x \in R^n$ *one has*:

(i) $h(x) - 1 \leqslant o(\det \sigma_x) \leqslant 2n(h(x)+1)+1$.

(ii) *For every* $p \in N$, $0 < T \leqslant 1$, $0 < u$ *one has*

$$E(1/\det \sigma_x(t)^p)^{1/p} \geqslant \lambda_x / T^{(h(x)-1-u)/2},$$

where

$$\lambda_x = 1/cK^{c'}_{2n(h(x)+1)+3}(\varphi)\, 2^{c'' K_0(\varphi)},$$

c, c', *and* c'' *being constants depending on* $h(x)$, n, *and* p *but not on* φ_j, $0 \leqslant j \leqslant d$.

(iii) *The following assertions are equivalent*:

(a) $\dim \mathscr{D}_{\infty, x} = n$.

(b) $\det_x \in \bigcup_{k=0}^{\infty} \mathscr{S}_k$.

(c) $o(\det \sigma_x) < \infty$.

(d) *There is some* $k \in N$ *such that for every* $p \in N$,

$$\lim_{T \to 0} T^k E(1/\det \sigma_x(T)^p)^{1/p} = 0.$$

(e) *There are some* $k, p \in N$ *such that*

$$\lim_{T \to 0} T^k E(1/\det \sigma_x(T)^p)^{1/p} = 0.$$

Proof. Since Theorem 5.1 is nothing but a translation of Theorem 3.3 in this context, the only thing to be done is to evaluate $Q_{1,r}(\det \sigma_x)$. Since the proof is straightforward we shall sketch it only. For a process $F \in \mathscr{C}_\infty$ we denote

$$K_r(F) = \max_{p(\alpha) \leqslant r} (Pr_\alpha(F))^* (1), \qquad r \in N.$$

One has

$$K_r(\det \sigma_x) \leqslant c_r K_0(\varphi)^{2n} \max_{1 \leqslant i,j \leqslant n} K_r(Y_j^i)^{2n} \max_{1 \leqslant i,j \leqslant n} K_r(Z_j^i)^{2n},$$

where c_r is a constant depending on r only.

It follows that

$$Q_{p,r}(\det \sigma_x) \leqslant E(K_r(\det \sigma_x)^{2p\mu_r})$$
$$\leqslant c_{r,n,p} K_0(\varphi)^{2np\mu_r} \max_{1 \leqslant i,j \leqslant n} [E(K_r(Y_j^i)^{4np\mu_r}) + E(K_r(Z_j^i)^{4np\mu_r})].$$

In order to evaluate $E(K_r(Y_j^i)^q)$, $(E(K_r(Z_j^i)^q)$, $q = 4np\mu_r$, we notice that, for $\alpha = (\alpha_1, ..., \alpha_m)$,

$$Pr_\alpha(Z)(t) = Z(t)\, T_{(\alpha_2, ..., \alpha_m)}(\varphi_{\alpha_1})(t, X(t)).$$

This is a consequence of (4.9). Then

$$E(Pr_\alpha^*(Z_j^i)(1)^q) \leqslant c_{n,r} K_r(\varphi)^q\, E((Z_j^i)^*\,(1)^q).$$

A classical argument based on Eq. (4.4) yields

$$E((Z_j^i)^*\,(1)^q) \leqslant 2^{c_q K_0(\varphi)}$$

and consequently

$$E(K_r(Z_j^i)^q) \leqslant c_{n,r,q} K_r(\varphi)^q\, 2^{c_q K_0(\varphi)}.$$

The same inequality holds for Y instead of Z. By using the above inequality for $q = 4np\mu_r$ one concludes that

$$Q_{p,r}(\det \sigma_x) \leqslant c_{n,r,p} K_r(\varphi)^{8np\mu_r}\, 2^{c'_{n,r,p} K_0(\varphi)}.$$

This, together with Theorem 3.3(ii), finishes to prove point (ii) in Theorem 5.1. Q.E.D.

To finish we make some comments.

Remark. 5.2. (i) Let us notice that if (iii)(b) in Theorem 5.1 holds then

$$E(|\det \sigma_x(T)|^{-p}) < \infty \qquad \text{for every} \quad p, T > 0. \tag{5.11}$$

To check it one notices that the above assertion is equivalent to $E(|\det(\int U_x \hat{U}_x)(T)|^{-p}) < \infty$ for every $p, T > 0$. In view of (5.6), $o(\det(\int U_x \hat{U}_x)) = o(\det \sigma_x) < \infty$ and consequently $\det(\int U_x \hat{U}_x) \in \mathscr{S}_{k/2}$ for some $k \in N$ (see Theorem 3.3). Then $1/(\det(\int U_x \hat{U}_x))^*\,(T)$ has finite moments of any order (see Remark 3.4). Finally, since $t \to \det(\int U_x \hat{U}_x)(t)$ is non-decreasing, the proof of (5.11) is completed.

As is well known (5.11) represents the assumption needed in the Malliavin absolute continuity theorem in order to get a smooth density $p(t, x, y)$ (with respect to the Lebesgue measure) for the semigroups of the diffusion X (see any of [1, 4, 5], ...). We also recall that (iii)(a) in Theorem 5.1 represents Hörmander's condition and the distance between it and (5.11) has already been discussed in Remark 3.4.

(ii) Although in Theorem 4.1(ii) we have given both lower and upper bounds for $E(\Delta_U(T)^{-p})^{1/p}$, in Theorem 5.1(ii) we abandon the upper bounds, because the instruments in our paper do not allow us to evaluate $H_{p,k}(\det \sigma_x)$ in terms of $K_k(\varphi)$ (the only thing we know is that it is finite), and so we would obtain an evaluation which is less precise than that already given by Kusuoka and Stroock in Corollary 3.25 in [4]. So only the lower bound is really new and interesting here.

Kusuoka and Stroock use the upper bounds for $E(\det \sigma_x(T)^{-p})$ to get upper bounds for $p(t, x, y)$ (the density of the semigroup) and for its derivatives. This is a rather straightforward application of Malliavin's differential calculus. One would be tempted to do the same thing for the lower bounds but it is not clear to us how this would be achieved. Let us notice that in [7], Kusuoka and Stroock obtain lower bounds for $p(t, x, y)$ but this is done under some more restrictive assumptions on the coefficients of the differential operator.

(iii) In [2] Fefferman and Phong show that Hörmander's condition is necessary and sufficient for subellipticity, so the result in our paper is a probabilistic counterpart of their result. Although it is to be expected that some connection exists between Theorem 5.1 here and the subelliptic estimates in [2], we are not able to make it explicit.

(iv) Many of the authors who dealt with the hypoellipticity problem from the probabilistic point of view have considered Markov homogeneous diffusions only, so Theorem 5.1 represents progress in this sense. Nevertheless non-homogeneous diffusions have been studied by Kusuoka and Stroock (see Example (3.14) in [5]) but under rather different hypotheses: they do not require smoothness for $t \to \varphi_j(t, x)$, $0 \leqslant j \leqslant d$ (only measurability is needed there), but assume that a strong ellipticity hypothesis holds. (Non-homogeneous diffusions are studied by Kusouka and Stroock in the frame of non-Markov stochastic equations.) On the other hand Ichihara and Kunita [3] used the analytic method (i.e., Hörmander's theorem) to get sufficient conditions in order that the semigroup of a diffusion process admits a smooth density with respect to the Lebesgue measure. In the frame there the non-homogeneous diffusions fit in the case of the parabolic operators. The result presented in Theorem 5.1 here is a little bit more precise than the one in [3] in the sense that we require Hörmander's condition to hold at a point x only, while they have to assume that it holds for any x.

(v) The strong boundedness assumption we required for the coefficients φ_j, $0 \leqslant j \leqslant d$, is not essential except for the evaluation of $Q_{p,r}(\det \sigma_x)$. Even this evaluation is possible for coefficients with polynomial growth but it involves a more careful calculation. The instruments needed can be found in Section 1 in Kusuoka and Stroock [6].

REFERENCES

1. J. M. BISMUT, Martingales, the Malliavin calculus and hypoellipticity under general Hormander's conditions, *Z. Wahrsch.* **56** (1981), 469–505.
2. C. FEFFERMAN AND D. PHONG, Subelliptic eigenvalue problems, *in* "Conference on Harmonic Analysis in Honor of A. Zygmund" (W. Beekner et al., Eds.), Wadsworth International Math. Series, Wadsworth, Belmont, CA, 1981.
3. K. ICHIHARA AND H. KUNITA, A Classification of the Second Order Degenerate Elliptic Operators and its Probabilistic Characterization, *Z. Wahrsch. Gebiete* **30** (1974), 235–254.
4. N. IKEDA AND S. WATANABE, "Stochastic Differential Equations and Diffusion Processes," North-Holland/Kodansha, Amsterdam/Tokyo, 1981.
5. S. KUSUOKA AND D. STROOCK, Applications of the Malliavin calculus I, Stochastic Analysis, *in* "Proc. Taniguchi Internal Symp. Katata and Kyoto, 1982" (K. Ito, Ed.), pp. 271–306, Kinokuniya, Tokyo, 1984.
6. S. KUSUOKA AND D. STROOCK, Applications of the Malliavin calculus, II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985), 1–76.
7. S. KUSUOKA AND D. STROOCK, Applications of the Malliavin calculus, III, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34**, No. 2 (1987), 391–142.
8. P. MALLIAVIN, Stochastic calculus of variation and hypoelliptic operators, *in* "Proc. Int. Symp. on SDE Kyoto" (K. Ito, Ed.), pp. 195–263, Kinokuniya, Tokyo, 1978.
9. S. WATANABE, "Stochastic Differential Equations and Malliavin Calculus," Tata Institute of Fundmental Research, Springer-Verlag, Berlin/Heidelberg//New York/Tokyo, 1984.
10. M. ZAKAI, The Malliavin calculus, *Acta Appl. Math.* **3** (1985), 175–207.